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Asymptotics for the Weyl quantized phase

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Abstract. In a previous paper, we considered Weyl quantization of functions of the angle in phase space, in particular a phase operator $\Delta(\varphi)$ and the quantized exponentials $\Delta(\mathrm{e}^{\pm i\varphi})$. In this paper we consider the first and second moments of these operators with respect to the harmonic oscillator Hermite states h_n and the coherent states Φ_α. Taking asymptotic limits we find, for example, that

$$\operatorname{var}[\Delta(\varphi); h_n] = \frac{\pi^2}{3} + O\left(\frac{\log n}{n}\right) \qquad (n \to \infty)$$

for the variance of $\Delta(\varphi)$ in the Hermite states. For the second moment of the phase operator in the coherent states we obtain the asymptotic limit

$$\|[\Delta(\varphi) - \theta]\Phi_\alpha\|^2 = O\left(\frac{1}{|\alpha|}\right)$$

as $|\alpha|$ tends to infinity, amongst other results.

1. Introduction

The search for a sensible quantization of phase is compelling both as a fundamental problem in quantum mechanics and as an application of that theory to the physics of cavity fields [1, 21]. We are aware of three distinct current theories of phase quantization. The first is that of Garrison and Wong [4], later also considered by Popov and Yarunin [5], Galindo [6], Grabowski [7], and others. The second is that of Barnett and Pegg and their collaborators [15–17]. The third is the Wigner–Weyl quantization, $\Delta(\varphi)$, of phase angle given by us in [1, 2], and independently considered by Royer [3].

We find that the physically interesting problem of the quantization of phase and its functions can be mathematically delicate: phase quantization bears the quantum hallmark. In this paper, as a further contribution to the subject, we present some rigorous results about $\Delta(\varphi)$ and the quantizations $\Delta(\mathrm{e}^{\pm i\varphi})$ of the complex exponentials of the phase. The burden of a subsequent paper [19] is then to extend the analysis to the question of the measurement of these quantites, allowing a comparison of important aspects of the three theories of phase.

In [1] and [2] we expressed $\Delta(\varphi)$ in terms of its matrix coefficients with respect to the standard Hermite basis for $L^2(\mathbb{R})$ and in [2] we also expressed it in terms of an integral kernel. Royer [3] *has considered the quantization of* φ *in a number of quantization orderings* other than that of Weyl.

In some sense $\Delta(\varphi)$ can be seen as a deformation of the Garrison and Wong operator [4], which we shall denote by X. Garrison and Wong obtained X as the angle function on the Hardy space $H^2(\mathbb{T})$ on the unit circle. It turns out that X is a Toeplitz operator, and so we shall refer to it as the Toeplitz phase operator hereafter. It can also be obtained by attempting to achieve canonicity with the number operator by doubling up the Hilbert space, as was done by Rocca and Sirugue [8], Lévy-Leblond [9], Newton [10], Ban [11], and others. The technique is equivalent to the Naimark extension theory of dilations and compressions, and when the compression back to the original Hilbert space is determined—as is required by the precepts of quantum mechanics—the operator X results. The extension method, then, results in nothing new.

Garrison and Wong [4] considered X and the number operator as acting on a special domain in Hardy space on which they are canonical. Unfortunately, the special domain is not invariant under any of the other basic operators of quantum mechanics, so we can say that the canonicity is incompatible with the no-go theorem which says that in quantum mechanics no phase operator can be canonically conjugate to the number operator [12, 13].

It is our contention that since X is not the Weyl quantization of a function of the angle in phase space [2], our operator $\Delta(\varphi)$ has a more immediate physical significance for phase, and so its properties merit further study. In addition, we do not believe that the angle coordinate in Hardy space directly corresponds to the angle inherent in quantum phase, at least in a direct way.

Certain fairly detailed information concerning the operator X is available. For example, a complete spectral representation of X is given in Garrison and Wong [4]. But there are gaps in what we know about X. For example, an expression for the asymptotic form of the variance of X in the coherent state Φ_α is not known, although Garrison and Wong present an outline proof indicating that the variance tends to zero as $|\alpha|$ tends to infinity. Detailed information along these lines has not been available for $\Delta(\varphi)$ until now, and it is the aim of this paper to remedy this.

A serious problem in this field is that it is not clear in operational terms exactly what physical observable a given operator represents. Conversely, it is not clear what operator will represent those effects which have been measured to date. In particular, it seems not to be known whether any of the experiments have measured some quantized angle directly, or whether they have measured some function of it, such as its cosine, sine or complex exponential. From a quantum mechanical point of view this makes a significant difference. For instance, the quantization $\Delta(\varphi)$ of the phase angle and the quantization $\Delta(\mathrm{e}^{\pm i\varphi})$ are not closely related. In fact, we know that

$$\Delta(\mathrm{e}^{\pm i\varphi}) \neq \mathrm{e}^{\pm i\Delta(\varphi)}. \tag{1.1}$$

Indeed, the operators $\Delta(\mathrm{e}^{\pm i\varphi})$ are not unitary [2], but the operators $\mathrm{e}^{\pm i\Delta(\varphi)}$ are.

Expressing the difference between them in terms of phase space functions, $\Delta(\mathrm{e}^{\pm i\varphi})$ is the Weyl quantization of the phase space function

$$\sum_{n=0}^{\infty} \frac{(\pm i\varphi)^n}{n!} \tag{1.2a}$$

whereas $\mathrm{e}^{\pm i\Delta(\varphi)}$ is the Weyl quantization of the phase space function

$$\sum_{n=0}^{\infty} \frac{(\pm i)^n}{n!} \underbrace{\varphi * \varphi * \cdots * \varphi}_{n \text{ times}} \tag{1.2b}$$

where $f * g$ indicates the Moyal product, which is the transferrence of the operator product on Hilbert space to phase space [14].

Evidently, it is of cardinal importance to relate the theoretical and experimental results in this field. As part of this process, it is necessary to have reliable mathematical analysis for all the proposed phase operators and the basic trigonometric functions of them. Since the mathematically predicted results are, typically, quite distinct for the different proposals, this analysis should enable us to determine what observables the experiments are actually measuring.

In a different direction, Barnett, Pegg, Vaccaro and their collaborators, e.g. [15–17] have proposed certain objects as 'a phase operator' and various 'states of definite phase'. They believe that they are proposing a generalization of quantum theory [18], but this cannot be accepted until and unless they prove that their formalism subsumes all the phenomena that quantum mechanics is able to describe. In a companion paper [19] we shall consider their formalism at length.

In this paper, we shall present various results concerning the expectation and variance of the operators $\Delta(\varphi)$ and $\Delta(\mathrm{e}^{\pm i\varphi})$ when acting on eigenstates of the harmonic oscillator and on coherent states. In particular, we have found their asymptotic behaviour for large index n of the Hermite functions h_n, and for large values $|\alpha|$ of the coherent state parameter. Obtaining these results turns out to be a matter of some surprising technical complexity, which seems to be a feature of careful angular quantization.

Angular quantization entails a choice of polar angle in the phase plane. This requires a choice of fiducial, or reference, angle θ_0, a choice that must also be made in any work on the phase operator. For purposes of comparison, we note that our phase plane angle φ is the complement of the angle used in some other papers on this subject, and our reference angle is taken to be $-\pi$, which corresponds to using the principal branch of the arctangent function to define the angle in phase space. The results obtained from different choices of θ_0 are easily related to one another [1,2].

This and the succeeding paper on approximation theory [19] continues our programme of examining the properties of a phase operator which is consistent with quantum mechanics, arising as it does from Weyl quantization of the angle in phase space.

At this point we recall the precise form of Wigner–Weyl quantization as we mean it and shall use it below. We write Π for $\mathbb{R}^2$ interpreted as phase space, and suppose that $T \in \mathcal{S}(\Pi)'$ is a tempered distribution. In order to be able to consider quantization of such singular objects we must utilize the integral transform method. That is, we first define the Wigner transform as the map $\mathcal{G} : \mathcal{S}(\mathbb{R}^2) \to \mathcal{S}(\Pi)$ given by

$$\mathcal{G}(g \otimes f)(p,q) = \frac{1}{2\pi} \int \infty_{-\infty} g(q + \tfrac{1}{2}x)\mathrm{e}^{\mathrm{i}px} f(q - \tfrac{1}{2}x)\,\mathrm{d}x. \tag{1.3}$$

The wavefunctions, f and g, are restricted to be test functions in Schwartz space at this point, where we adopt the convention $\hbar = 1$.

Since $\mathcal{G}(g \otimes f)$ is a test function in $\mathcal{S}(\Pi)$, any tempered distribution can safely act on it, and so the equation

$$(\Delta[T]f)(g) = T(\mathcal{G}(g \otimes f)) \tag{1.4}$$

is well defined, and serves to define the quantization $\Delta[T]$ as a continuous linear map from $\mathcal{S}(\mathbb{R})$ into $\mathcal{S}(\mathbb{R})'$.

The purpose of this formulation is to use the fact that we can safely restrict f and g to be test functions—$\mathcal{S}(\mathbb{R})$ is dense in $L^2(\mathbb{R})$—and use the freedom gained to balance the singular nature of T. The price paid, aside from the indirect nature of the expressions, is that $\Delta[T]$ is not a Hilbert space operator unless T is regular enough.

Unfortunately we know of no useful general regularity condition which will guarantee this, but in most cases of physical interest, more or less practicable and effective conditions are known. Interestingly, quantization of angle functions sits more or less at the critical point, which is part of the reason this subject is technically difficult.

The bilinear pairing used in distribution theory does not involve the complex conjugation which occurs in the inner product on $L^2(\mathbb{R})$. Our convention is to put the conjugation on the first element, and so when a distribution S happens to be an element of $L^2(\mathbb{R})$, we can write

$$S(f) = \langle \bar{f}, S\rangle \qquad S \in L^2(\mathbb{R}), f \in \mathcal{S}(\mathbb{R}).$$

Then if the distribution T is regular enough so that its quantization $\Delta(T)$ is no worse than an unbounded operator from $\mathcal{S}(\mathbb{R})$ to $L^2(\mathbb{R})$, we can write

$$[\Delta(T)f](g) = \langle \bar{g}, \Delta(T)f\rangle \qquad g \in \mathcal{S}(\mathbb{R}), f \in L^2(\mathbb{R}). \tag{1.5}$$

In particular (with the cut at $\theta_0 = -\pi$), the bounded phase operator $\Delta(\varphi)$ is obtained by quantizing the phase angle

$$\varphi(p,q) = \begin{cases} 2\arctan[(\sqrt{p^2+q^2}-p)/q] & \text{if } q \neq 0 \\ 0 & \text{if } q = 0 \text{ and } p \geqslant 0 \\ -\pi & \text{if } q = 0 \text{ and } p < 0 \end{cases} \tag{1.6}$$

in the interval $[-\pi, \pi)$. For further details along these lines, see [2].

In section 2 of this paper we re-express the action of $\Delta(\varphi)$ on the harmonic oscillator states h_n. This is needed in order to obtain rigorous results, in section 3, for the standard deviation of $\Delta(\varphi)$ with respect to these states (see equation (3.10)). Qualitatively, the variance straddles the value $\pi^2/3$, alternately for even and odd n, and converges to this value as n tends to infinity (see theorem 3.1).

Section 4 calls attention to certain polynomials $\psi_n(q)$ which are a by-product of this analysis; they appear to be a deformation of the Hermite polynomials, and satisfy inhomogeneous recurrence relations.

In section 5 we consider the means and standard deviations of the operators $\Delta(e^{\pm i\varphi})$ with respect to coherent states Φ_α. We obtain an exact expression for means in terms of Bessel functions (equation (5.9)), and so can find its asymptotic form for large $|\alpha|$ (equation (5.10)). We then prove that the variance of $\Delta(e^{\pm i\varphi})$ behaves like $1/(2|\alpha|^2)$ for large $|\alpha|$ (equation (5.13)).

In section 6 we look at the mean and variance of $\Delta(\varphi)$ for the states Φ_α. For large $|\alpha|$, the variance of $\Delta(\varphi)$ goes like $|\alpha|^{-1}$. This is a subtle result, for if the infinite series representing the variance, obtained from the Hermite function expansion for Φ_α, were truncated *at any term*, the (truncated) variance would have an $|\alpha|^{-2}$ asymptotic leading term. A by-product of the analysis in section 6 is a sharpening of our knowledge of the spectrum of $\Delta(\varphi)$, equation (6.32): its spectrum contains the continuous interval $[-\pi, \pi]$. This result is consistent with our belief, based partly on earlier computer work [2], that the spectrum of $\Delta(\varphi)$ is absolutely continuous and equal to $[-\pi, \pi]$.

2. Preliminary results

Starting from the kernel expression for $\Delta(\varphi)$ we shall obtain an expression for $\Delta(\varphi)$ in this section, as the difference of two unbounded operators, and use the result to determine a expression for $\Delta(\varphi)h_n$. This will enable us to determine various asymptotic expressions in later sections.

As was shown in [2], the operator $\Delta(\varphi)$ has the integral kernel expression

$$[\Delta(\varphi)g](q) = \frac{\pi}{2}\operatorname{sign}(q)g(q) - \frac{i}{2}(\mathcal{Z}g)(q) \qquad g \in \mathcal{S}(\mathbb{R}) \tag{2.1}$$

where

$$(\mathcal{Z}g)(q) = \mathrm{Pv}\int_{\mathbb{R}} \mathrm{sign}(p+q)\frac{1}{q-p}\mathrm{e}^{-\frac{1}{2}|p^2-q^2|}g(p)\,\mathrm{d}p. \tag{2.2}$$

Applying some elementary manipulations to this definition, we find that

$$\begin{aligned}(\mathcal{Z}g)(q) = \int_0^{2q} \mathrm{e}^{-qt}[\mathrm{e}^{\frac{1}{2}t^2}g(q-t) - \mathrm{e}^{-\frac{1}{2}t^2}g(q+t)]t^{-1}\,\mathrm{d}t \\ - \int_{2q}^{\infty} \mathrm{e}^{-\frac{1}{2}t^2}[\mathrm{e}^{qt}g(q-t) + \mathrm{e}^{-qt}g(q+t)]t^{-1}\,\mathrm{d}t\end{aligned} \tag{2.3}$$

for $g \in \mathcal{S}(\mathbb{R})$ and $q > 0$.

If we define the argument reversal operator on $\mathcal{S}(\mathbb{R})$,

$$\mathcal{R}g(q) = g(-q) \qquad q \in \mathbb{R},\ g \in \mathcal{S}(\mathbb{R}) \tag{2.4}$$

then $\mathcal{Z}$ and $\mathcal{R}$ commute:

$$\mathcal{Z}\mathcal{R}g = \mathcal{R}\mathcal{Z}g \qquad g \in \mathcal{S}(\mathbb{R}). \tag{2.5}$$

Applying this equality to our expression for $\mathcal{Z}g$ yields an analogous expression valid for $q < 0$.

Fundamental to many of our calculations will be the coherent states. For any complex number α, we define the unit vector

$$\Phi_\alpha = \mathrm{e}^{-\frac{1}{4}|\alpha|^2}\sum_{n\geqslant 0}\frac{\mathrm{i}^n\bar{\alpha}^n}{\sqrt{2^n n!}}h_n \tag{2.6}$$

where $\{h_n : n \geqslant 0\}$ is, as usual, the standard orthonormal basis for $L^2(\mathbb{R})$, consisting of eigenstates of the harmonic oscillator. As is very well known, Φ_α is a translated Gaussian, with values

$$\Phi_\alpha(q) = \frac{1}{\pi^{1/4}}\exp[\tfrac{1}{4}(\bar{\alpha}^2 - |\alpha|^2)]\exp[-\tfrac{1}{2}q^2 + \mathrm{i}\bar{\alpha}q]. \tag{2.7}$$

Note that our definition of coherent states differs slightly from that found elsewhere; what we have described as being parametrized by α would be described elsewhere as being parametrized by $\mathrm{i}\bar{\alpha}/\sqrt{2}$. The reasons for this change partly relate to the complementary value of our angle φ to the choice found elsewhere, and partly to calculational convenience.

Later on we shall be concerned with the states Φ_α in their own right, but for the present we shall restrict ourselves to the cases where α is purely imaginary, and write

$$\Psi_\beta = \Phi_{\mathrm{i}\beta} = \mathrm{e}^{-\frac{1}{4}\beta^2}\sum_{n\geqslant 0}\frac{\beta^n}{\sqrt{2^n n!}}h_n \tag{2.8}$$

so that Ψ_β is $\mathrm{e}^{-\frac{1}{4}\beta^2}$ times the standard generating function, G_β, for the $\{h_n : n \geqslant 0\}$. Thus

$$\Psi_\beta(q) = \frac{1}{\pi^{1/4}}\exp[-\tfrac{1}{2}(q-\beta)^2] \tag{2.9}$$

for any $\beta \in \mathbb{R}$. Direct calculation shows us that

$$[\mathcal{Z}\Psi_\beta](q) = \Psi_\beta(q)G(\beta, q) \qquad q > 0$$

where

$$G(\beta, q) = \int_0^{2q}(\mathrm{e}^{-\beta t} - \mathrm{e}^{-t^2-2qt+\beta t})t^{-1}\,\mathrm{d}t - 2\int_{2q}^{\infty}\mathrm{e}^{-t^2}\cosh[(2q-\beta)t]t^{-1}\,\mathrm{d}t. \tag{2.10}$$

Our first aim is to obtain information about the function G. To begin with, if we set $\beta = 0$ and differentiate with respect to q we obtain the equation

$$\frac{\mathrm{d}}{\mathrm{d}q}G(0,q) = \frac{2}{q} \qquad q > 0$$

so that

$$G(0,q) = 2(\log q + k) \qquad q > 0$$

for some constant k. It will turn out that this log term represents a singular part of the action of $\Delta(\varphi)$.

We may substitute this into the expression for $\mathcal{Z}$ acting on h_0, obtaining

$$[\mathcal{Z}h_0](q) = [\mathcal{Z}\Psi_0](q) = G(0,q)\Psi_0(q) = 2(\log q + k)h_0(q) \qquad q > 0.$$

Since $\mathcal{R}h_0 = h_0$, it follows that

$$[\mathcal{Z}h_0](q) = 2(\log(|q|) + k)h_0(q) \qquad q \neq 0.$$

If we now substitute this into $\Delta(\varphi)h_0$, the only unknown is the constant k:

$$[\Delta(\varphi)h_0](q) = \left[\frac{\pi}{2}\operatorname{sign}(q) - \mathrm{i}(\log(|q|) + k)\right]h_0(q) \qquad q \neq 0.$$

Taking the inner product of this expression with h_0 now enables us to evaluate k; since we know that

$$\langle h_0, \Delta(\varphi)h_0\rangle = 0$$

from [1], we have

$$\begin{aligned} 0 &= \frac{1}{\pi^{1/2}}\int_R \left[\frac{\pi}{2}\operatorname{sign}(q) - \mathrm{i}(\log(|q|) + k)\right]\mathrm{e}^{-q^2}\,\mathrm{d}q \\ &= \frac{-2\mathrm{i}}{\pi^{1/2}}\int_0^\infty (\log q + k)\mathrm{e}^{-q^2}\,\mathrm{d}q \\ &= \frac{-2\mathrm{i}}{\pi^{1/2}}[\tfrac{1}{2}\pi^{1/2}k - \tfrac{1}{4}\pi^{1/2}(\gamma + 2\log 2)] \end{aligned}$$

so that

$$k = \log 2 + \tfrac{1}{2}\gamma$$

where γ is the Euler–Mascheroni constant.

Thus we have shown that

$$G(0,q) = 2\log(2q) + \gamma \qquad q > 0 \tag{2.11}$$

and

$$[\Delta(\varphi)h_0](q) = [\tfrac{1}{2}\pi\operatorname{sign}(q) - \mathrm{i}\log(2|q|) - \mathrm{i}\tfrac{1}{2}\gamma]h_0(q) \qquad q \neq 0 \tag{2.12}$$

which is the first of the results we shall use in subsequent sections.

Moving on to consider the case for general β, direct calculation shows that

$$\frac{\partial}{\partial\beta}G(\beta,q) = \beta^{-1}(\mathrm{e}^{-2\beta q} - 1) + \mathrm{e}^{q^2}\int_q^\infty \mathrm{e}^{-t^2-\beta(q+t)}\,\mathrm{d}t - \mathrm{e}^{q^2}\int_q^\infty \mathrm{e}^{-t^2-\beta(q-t)}\,\mathrm{d}t$$

and so

$$G(\beta,q)-G(0,q)=\int_0^\beta \xi^{-1}(e^{-2\xi q}-1)\,d\xi+e^{q^2}\int_q^\infty e^{-t^2}(1-e^{-\beta(q+t)})(q+t)^{-1}\,dt$$
$$-e^{q^2}\int_q^\infty e^{-t^2}(1-e^{-\beta(q-t)})(q-t)^{-1}\,dt$$
$$=\int_0^\beta \xi^{-1}(e^{-2\xi q}-1)\,d\xi-\sum_{n\geqslant 1}\frac{(-\beta)^n e^{q^2}}{n!}\int_q^\infty e^{-t^2}[(q+t)^{n-1}-(q-t)^{n-1}]\,dt$$
$$=\sum_{n\geqslant 1}\frac{(-\beta)^n}{n!}\left[\frac{(2q)^n}{n}-\pi_{n-1}(q)\right] \qquad (2.13a)$$

where

$$\pi_n(q)=e^{q^2}\int_q^\infty e^{-t^2}[(q+t)^n-(q-t)^n]\,dt \qquad (2.13b)$$

for $n\geqslant 0$.

We can obtain recurrence relations for the functions π_n from the integral representation:

$$\pi_n'(q)=2q\pi_n(q)+n\pi_{n-1}(q)-(2q)^n \qquad n\geqslant 1 \qquad (2.14a)$$
$$\pi_n(q)=q\pi_{n-1}(q)+\tfrac{1}{2}(n-1)\pi_{n-2}(q)+\tfrac{1}{2}(2q)^{n-1} \qquad n\geqslant 2 \qquad (2.14b)$$

from which follows

$$\pi_n(q)=\tfrac{1}{2}\pi_{n-1}'(q)+(2q)^{n-1} \qquad n\geqslant 1 \qquad (2.14c)$$

while

$$\pi_0(q)=0. \qquad (2.14d)$$

From this it is straightforward to deduce that $\pi_n(q)$ is a polynomial of degree $n-1$ with parity $(-1)^{n-1}$ and leading coefficient 2^{n-1} for any $n\in\mathbb{N}$. Indeed,

$$\pi_{2n+1}(q)=\sum_{j=0}^{n}\frac{(n+j)!}{(2j)!}(2q)^{2j} \qquad n\geqslant 0 \qquad (2.15a)$$
$$\pi_{2n}(q)=\sum_{j=1}^{n}\frac{(n+j)!}{(2j+1)!}(2q)^{2j+1} \qquad n\geqslant 1. \qquad (2.15b)$$

Since

$$[\mathcal{Z}\Psi_\beta](-q)=[\mathcal{Z}\mathcal{R}\Psi_\beta](q)=[\mathcal{Z}\Psi_{-\beta}](q)$$
$$=G(-\beta,q)\Psi_{-\beta}(q)=G(-\beta,q)\Psi_\beta(-q)$$

it follows that for $q\neq 0$,

$$[\mathcal{Z}\Psi_\beta](q)=\left[2\log(2|q|)+\gamma+\sum_{n\geqslant 1}\frac{(-\beta)^n}{n!}\left\{\frac{(2q)^n}{n}-\pi_{n-1}(q)\right\}\right]\Psi_\beta(q). \qquad (2.16)$$

This expression enables us to rewrite equation (2.1) for $g=\Psi_\beta$ as follows:

$$[\Delta(\varphi)\Psi_\beta](q)=\left[\sum_{n\geqslant 0}\frac{(-\beta)^n}{n!}F_n(q)\right]\Psi_\beta(q) \qquad (2.17a)$$

where

$$F_n(q) = \begin{cases} \dfrac{\pi}{2}\,\mathrm{sign}(q) - \mathrm{i}\log(2|q|) - \dfrac{\mathrm{i}}{2}\gamma & \text{if } n = 0 \\ -\dfrac{\mathrm{i}}{2}\left[\dfrac{(2q)^n}{n} - \pi_{n-1}(q)\right] & \text{if } n \geqslant 1. \end{cases} \tag{2.17b}$$

We obtain a similar expression for $\Delta(\varphi)$ acting on h_n by substituting this into

$$\begin{aligned} [\Delta(\varphi)G_\beta](q) &= \left[\sum_{n\geqslant 0}\frac{(-\beta)^n}{n!}F_n(q)\right]G_\beta(q) \\ &= \left[\sum_{n\geqslant 0}\frac{(-\beta)^n}{n!}F_n(q)\right]\left[\sum_{n\geqslant 0}\frac{\beta^n}{\sqrt{2^n n!}}h_n(q)\right]. \end{aligned} \tag{2.18}$$

For by equating coefficients of equal powers of β we obtain

$$[\Delta(\varphi)h_n](q) = \sqrt{2^n n!}\sum_{m=0}^{n}\frac{(-1)^{n-m}}{(n-m)!\sqrt{2^m m!}}F_{n-m}(q)h_m(q) \qquad n \geqslant 0 \tag{2.19a}$$

from which we deduce that for $n \geqslant 0$,

$$[\Delta(\varphi)h_n](q) = \left[\frac{\pi}{2}\,\mathrm{sign}(q) - \mathrm{i}\log(2|q|) - \frac{\mathrm{i}}{2}\gamma\right]h_n(q) + \psi_n(q)\mathrm{e}^{-\frac{1}{2}q^2}. \tag{2.19b}$$

Here ψ_n is a certain polynomial in q of degree n and parity $(-1)^n$, of which more will be said in section 4. Thus we can find constants $X_{m,n}$ for $0 \leqslant m \leqslant n$ such that

$$[\Delta(\varphi)h_n](q) = \left[\frac{\pi}{2}\,\mathrm{sign}(q) - \mathrm{i}\log(2|q|) - \frac{\mathrm{i}}{2}\gamma\right]h_n(q) + \sum_{m\leqslant n}X_{m,n}h_m \tag{2.19c}$$

where, moreover, $X_{m,n} = 0$ whenever $n - m$ is odd.

This represents a substantial simplification of the expressions for $\Delta(\varphi)h_n$ obtained from its matrix elements ([1, 2], ibid) or the form implicit in equations (2.1) and (2.2). In the next section we shall identify the constants $X_{m,n}$, and find that in a certain sense, the major part of $\Delta(\varphi)$ is the operator

$$C(\varphi) = \frac{\pi}{2}\,\mathrm{sign}(Q) - \mathrm{i}\log(2Q) - \frac{\mathrm{i}}{2}\gamma I.$$

We do not yet have a completely closed expression for $\Delta(\varphi)$. It is clear that $C(\varphi)$ is unbounded, and so we find ourselves in the somewhat unsatisfactory position of expressing the bounded operator $\Delta(\varphi)$ as the sum of unbounded operators. Nonetheless, our present knowledge is still adequate to perform a number of interesting calculations.

3. $\Delta(\varphi)$ and the harmonic oscillator eigenstates

We now proceed to calculate the variance of $\Delta(\varphi)$ for each of the harmonic oscillator eigenstates h_n ($n \geqslant 0$). It should be noted that the diagonal matrix elements vanish:

$$\langle h_n, \Delta(\varphi)h_n\rangle = 0 \qquad n \geqslant 0. \tag{3.1}$$

Then the variance of $\Delta(\varphi)$ in the pure state represented by h_n is

$$\mathrm{var}[\Delta(\varphi); h_n] = \langle h_n, \Delta(\varphi)^2 h_n\rangle - \langle h_n, \Delta(\varphi)h_n\rangle^2 = \|\Delta(\varphi)h_n\|^2.$$

Calculations for the Toeplitz phase operator yield

$$\mathrm{var}[X; h_n] = \frac{\pi^2}{3} - \sum_{k=n+1}^{\infty}\frac{1}{k^2} \tag{3.2}$$

for its variance in the state h_n. As a sequence in n, this monotonically increases to $\pi^2/3$ as $n \to \infty$, which is consistent with a 'classical' random distribution of phase. The variance of $\Delta(\varphi)$ will not have this form and its convergence will no longer be monotonic, but we shall see that its limit as $n \to \infty$ is also $\pi^2/3$.

Parenthetically, we note that the Barnett and Pegg approach yields [17] a variance of $\pi^2/3$ for their 'phase operator' X_s in all number operator eigenstates h_n in the limit as s tends to infinity. This is not entirely surprising, since

$$\langle h_n, X_s^2 h_n\rangle = \frac{1}{s+1}\sum_{m=0}^{s}\theta_{s,m}^2 \qquad \text{where } \theta_{s,m} = \theta_0 + \frac{m}{s+1}2\pi \qquad (m = 0, 1, \ldots, s) \tag{3.3a}$$

and

$$X_s = \sum_{m=0}^{s}\theta_{s,m}P_{s,m} \tag{3.3b}$$

with $P_{s,m}$ the projection operator onto the state

$$\chi_s(\theta_{s,m}) = \frac{1}{\sqrt{s+1}}\sum_{n=0}^{s}\mathrm{e}^{\mathrm{i}n\theta_{s,m}}h_n. \tag{3.3c}$$

The limit $s \to \infty$ is analogous to, but not the same as, a Césaro mean, and the result is that only the terms with large n contribute. The complication is due to the fact that the $\theta_{s,m}$ depend on s as well as m.

Continuing the calculations begun in the previous section, define operators $\mathcal{A}$, $\mathcal{B} \in \mathcal{L}^+[\mathcal{S}(\mathbb{R}), L^2(\mathbb{R})]$ by setting

$$\mathcal{A} = \frac{\pi}{2}\operatorname{sign}(Q) \tag{3.4a}$$

$$\mathcal{B} = \log(2|Q|) + \tfrac{1}{2}\gamma I. \tag{3.4b}$$

Here Q is the usual position operator on $L^2(\mathbb{R})$. The set $\mathcal{L}^+[\mathcal{S}(\mathbb{R}), L(\mathbb{R})]$ consists of all continuous linear maps A from $\mathcal{S}(\mathbb{R})$ to $L^2(\mathbb{R})$ which have the following property: they have adjoints A^* which may be restricted to the domain $\mathcal{S}(\mathbb{R})$; and writing A^+ for this restriction, A^+ is also a continuous linear map from $\mathcal{S}(\mathbb{R})$ to $L^2(\mathbb{R})$. With this notation,

$$\mathcal{A} = \mathcal{A}^+ \qquad \text{and} \qquad \mathcal{B} = \mathcal{B}^+.$$

Thus

$$\Delta(\varphi)h_n = (\mathcal{A} - \mathrm{i}\mathcal{B})h_n + \sum_{m\leqslant n}X_{m,n}h_m$$

from which it follows that

$$X_{m,n} = \begin{cases} 2\mathrm{i}\langle h_m, \mathcal{B}h_n\rangle & \text{if } m < n \\ \mathrm{i}\langle h_m, \mathcal{B}h_n\rangle & \text{if } m = n \end{cases}$$

which leads us to the following expression for the matrix elements of $\Delta(\varphi)$ with respect to the Hermite functions:

$$\langle h_m, \Delta(\varphi)h_n\rangle = \begin{cases} \langle h_m, \mathcal{A}h_n\rangle + \mathrm{i}\langle h_m, \mathcal{B}h_n\rangle & \text{if } m < n \\ 0 & \text{if } m = n \\ \langle h_m, \mathcal{A}h_n\rangle - \mathrm{i}\langle h_m, \mathcal{B}h_n\rangle & \text{if } m > n. \end{cases} \tag{3.5}$$

Since $\langle h_m, \mathcal{A}h_n\rangle = 0$ unless $m - n$ is odd, while $\langle h_m, \mathcal{B}h_n\rangle = 0$ unless $m - n$ is even, we see here a very close relationship between $\mathcal{A}$, $\mathcal{B}$ and $\Delta(\varphi)$.

We can summarize our results so far by showing that

$$\Delta(\varphi)h_n = \mathcal{A}h_n - \mathrm{i}\mathcal{P}_n\mathcal{B}h_n \qquad n \geqslant 0 \tag{3.6a}$$

where $\mathcal{P}_n = \mathcal{P}_n^*$ is the bounded operator on $L^2(\mathbb{R})$ given by the rule

$$\mathcal{P}_n h_m = \begin{cases} -h_m & \text{if } m < n \\ 0 & \text{if } m = n \\ +h_m & \text{if } m > n. \end{cases} \tag{3.6b}$$

Moreover, it is clear that $\mathcal{A}h_n$ and $\mathcal{P}_n\mathcal{B}h_n$ are real-valued functions, so we deduce that

$$\begin{aligned} \mathrm{var}[\Delta(\varphi); h_n] &= \|\Delta(\varphi)h_n\|^2 = \|\mathcal{A}h_n\|^2 + \|\mathcal{P}_n\mathcal{B}h_n\|^2 \\ &= \tfrac{1}{4}\pi^2 + \|\mathcal{B}h_n\|^2 - \langle h_n, \mathcal{B}h_n\rangle^2 \\ &= \tfrac{1}{4}\pi^2 + \mathrm{var}[\mathcal{B}; h_n] \end{aligned} \tag{3.7}$$

for any $n \geqslant 0$. Thus we must determine the variance of the operator $\mathcal{B}$ in the harmonic oscillator eigenstates.

Standard tables of Laplace transforms show us that

$$\frac{\log(\varepsilon + 2\mathrm{i}s)}{\varepsilon + 2\mathrm{i}s} = -\int_0^\infty (\gamma + \log t)\mathrm{e}^{-(\varepsilon+2\mathrm{i}s)t}\,\mathrm{d}t \qquad \varepsilon, s > 0$$

so that

$$\begin{aligned} \int_0^\infty & \frac{1}{\varepsilon^2 + 4s^2}\left[2\varepsilon s \arctan\left(\frac{2s}{\varepsilon}\right) - 4s^2 \log\left(\sqrt{\varepsilon^2 + 4s^2}\right)\right] h_n(s)^2\,\mathrm{d}s \\ &= \frac{1}{\pi^{1/2}2^n n!}\int_0^\infty 2s\left(\int_0^\infty [\gamma + \log(t)]\mathrm{e}^{-\varepsilon t}\sin(2st)\,\mathrm{d}t\right)\mathrm{e}^{-s^2}H_n(s)^2\,\mathrm{d}s \\ &= \frac{1}{\pi^{1/2}2^n n!}\int_0^\infty [\gamma + \log(t)]\mathrm{e}^{-\varepsilon t}\left(\int_0^\infty 2s\mathrm{e}^{-s^2}H_n(s)^2\sin(2st)\,\mathrm{d}s\right)\mathrm{d}t \\ &= \int_0^\infty [\gamma + \log(t)]\mathrm{e}^{-\varepsilon t}t\mathrm{e}^{-t^2}[L_n(2t^2) - 2L_n'(2t^2)]\,\mathrm{d}t \end{aligned}$$

for any $\varepsilon > 0$. By H_n we mean the nth Hermite polynomial, and L_n is the nth Laguerre polynomial.

Letting $\varepsilon \to 0+$, we deduce that

$$\begin{aligned} -\int_0^\infty \log(2s)h_n(s)^2\,\mathrm{d}s &= \int_0^\infty [\gamma + \log(t)]t\mathrm{e}^{-t^2}[L_n(2t^2) - 2L_n'(2t^2)]\,\mathrm{d}t \\ &= -\tfrac{1}{2}\int_0^\infty [\gamma + \tfrac{1}{2}\log(u)]\frac{\mathrm{d}}{\mathrm{d}u}[L_n(2u)\mathrm{e}^{-u}]\,\mathrm{d}u \end{aligned}$$

so that the diagonal matrix elements of $\mathcal{B}$ have the integral representation

$$\begin{aligned} \langle h_n, \mathcal{B}h_n\rangle &= \tfrac{1}{2}\gamma + \int_R \log(2|s|)h_n(s)^2\,\mathrm{d}s \\ &= \tfrac{1}{2}\gamma + \int_0^\infty [\gamma + \tfrac{1}{2}\log(u)]\frac{\mathrm{d}}{\mathrm{d}u}[L_n(2u)\mathrm{e}^{-u}]\,\mathrm{d}u \\ &= \tfrac{1}{2}\int_0^\infty \log(u)\frac{\mathrm{d}}{\mathrm{d}u}[L_n(2u)\mathrm{e}^{-u}]\,\mathrm{d}u - \tfrac{1}{2}\gamma \end{aligned}$$

for any $n \geqslant 0$. Thus

$$\sum_{n\geqslant 0}\langle h_n, \mathcal{B}h_n\rangle \xi^n = \tfrac{1}{2}(1-\xi)^{-1}\int_0^\infty \log(u)\frac{\mathrm{d}}{\mathrm{d}u}\exp\left[-\left(\frac{1+\xi}{1-\xi}\right)u\right]\mathrm{d}u - \tfrac{1}{2}\gamma(1-\xi)^{-1}$$

$$= -\tfrac{1}{2}(1-\xi)^{-1}\log\left(\frac{1-\xi}{1+\xi}\right)$$

$$= \left(\sum_{n\geqslant 0}\xi^n\right)\left(\sum_{n\geqslant 0}\frac{\xi^{2n+1}}{2n+1}\right)$$

for $|\xi| < 1$, so we come to

$$\langle h_n, \mathcal{B}h_n\rangle = \sum_{m=0}^{[(n-1)/2]}\frac{1}{2m+1} \qquad n \geqslant 0. \tag{3.8}$$

We note that while the off-diagonal matrix elements of $\mathcal{B}$ are essentially matrix elements of $\Delta(\varphi)$, its diagonal elements are unbounded, with

$$\langle h_n, \mathcal{B}h_n\rangle = O(\log n) \qquad n \to \infty.$$

Of course this reflects the unbounded nature of $\mathcal{B}$.

An argument analogous to the preceding one shows us that

$$\int_0^\infty [\log(2s)]^2 h_n(s)^2\,\mathrm{d}s = \frac{\pi^2}{24} - \tfrac{1}{2}\int_0^\infty [\gamma + \tfrac{1}{2}\log(u)]^2\frac{\mathrm{d}}{\mathrm{d}u}[L_n(2u)\mathrm{e}^{-u}]\,\mathrm{d}u$$

and hence that

$$\|\mathcal{B}h_n\|^2 = \frac{\pi^2}{12} - \gamma\langle h_n, \mathcal{B}h_n\rangle - \frac{\gamma^2}{4} - \tfrac{1}{4}\int_0^\infty [\log(u)]^2\frac{\mathrm{d}}{\mathrm{d}u}[L_n(2u)\mathrm{e}^{-u}]\,\mathrm{d}u.$$

It follows from this that

$$\sum_{n\geqslant 0}\|\mathcal{B}h_n\|^2\xi^n = (1-\xi)^{-1}\left\{\frac{\pi^2}{8} + \frac{1}{4}\left[\log\left(\frac{1-\xi}{1+\xi}\right)\right]^2\right\}$$

for all $|\xi| < 1$. Then

$$\|\mathcal{B}h_n\|^2 = \frac{\pi^2}{8} + \sum_{\substack{0\leqslant l,m\leqslant[(n-2)/2]\\ l+m\leqslant[(n-2)/2]}}\frac{1}{(2l+1)(2m+1)} \qquad n \geqslant 0. \tag{3.9}$$

This gives us the variance of $\Delta(\varphi)$:

$$\operatorname{var}[\Delta(\varphi); h_n] = \frac{3\pi^2}{8} + \sum_{\substack{0\leqslant l,m\leqslant[(n-2)/2]\\ l+m\leqslant[(n-2)/2]}}\frac{1}{(2l+1)(2m+1)} - \left(\sum_{m=0}^{[(n-1)/2]}\frac{1}{2m+1}\right)^2 \tag{3.10}$$

for any $n \geqslant 0$. In particular,

$$\operatorname{var}[\Delta(\varphi); h_{2n}] = \frac{3\pi^2}{8} - \sum_{\substack{0\leqslant l,m\leqslant n-1\\ l+m>n-1}}\frac{1}{(2l+1)(2m+1)} \tag{3.11a}$$

and

$$\operatorname{var}[\Delta(\varphi); h_{2n+1}] = \frac{3\pi^2}{8} - \sum_{\substack{0\leqslant l,m\leqslant n\\ l+m\geqslant n}}\frac{1}{(2l+1)(2m+1)} \tag{3.11b}$$

for all $n \geqslant 0$.

From these formulae, all the results in the following theorem are easy to establish except that concerning the limit to $\pi^2/3$. This latter result is established by using Riemann integration techniques to express the variance as, e.g.,

$$\lim_{n\to\infty} \operatorname{var}[\Delta(\varphi); h_{2n}] = \frac{3\pi^2}{8} + \frac{1}{4}\int_0^1 \frac{\log(1-x)}{x}\,\mathrm{d}x \tag{3.12}$$

from which the result is immediate. We omit the remainder of the proof.

Theorem 3.1.

(a) The sequence $(\operatorname{var}[\Delta(\varphi); h_{2n+1}])_{n\geqslant 0}$ is monotonically increasing, with limit $\pi^2/3$.

(b) The sequence $(\operatorname{var}[\Delta(\varphi); h_{2n}])_{n\geqslant 0}$ is monotonically decreasing, with limit $\pi^2/3$.

(c) The variance of $\Delta(\varphi)$ in the state h_n has the asymptotic order

$$\operatorname{var}[\Delta(\varphi); h_n] = \frac{\pi^2}{3} + O\left(\frac{\log n}{n}\right) \qquad n \to \infty.$$

The sequence $(\operatorname{var}[\Delta(\varphi); h_n])_{n\geqslant 0}$ has an oscillatory character, since its even and odd subsequences are monotonically decreasing and increasing, respectively. A similar oscillatory behaviour was found for $(\|\Delta(\mathrm{e}^{\mathrm{i}\varphi})h_n\|^2)_{n\geqslant 0}$ in [2]. We recall that our result there was that

$$\|\Delta(\mathrm{e}^{\mathrm{i}\varphi})h_n\|^2 = \frac{n+1}{2}\frac{\Gamma(\frac{1}{2}n+s_n)^2}{\Gamma(\frac{1}{2}n+\frac{1}{2}+s_n)^2} \tag{3.13}$$

for all $n \geqslant 0$, where

$$s_n = \begin{cases} \frac{1}{2} & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd.} \end{cases} \tag{3.14}$$

We noted there, and it is easy to check, that the even subsequence decreases and the odd subsequence increases, and the sequence converges to 1. Thus

$$\lim_{n\to\infty} \|\Delta(\varphi)h_n\|^2 = \tfrac{1}{3}\pi^2 \tag{3.15a}$$

$$\lim_{n\to\infty} \|\Delta(\mathrm{e}^{\mathrm{i}\varphi})h_n\|^2 = 1 \tag{3.15b}$$

which is consistent with a uniform distribution of phase.

4. The polynomials $\psi_n(q)$

In passing, we note a few results which provide an interesting insight into the polynomials $\psi_n(q)$ obtained in section 2. In some sense they are deformations of the Hermite polynomials $H_n(q)$, and it would be a matter of some interest to clarify their properties further.

We recall that for $n \geqslant 0$ and $q > 0$,

$$[\Xi h_n](q) = [2\log(2q) + \gamma]h_n(q) + 2\mathrm{i}\psi_n(q)\mathrm{e}^{-\frac{1}{2}q^2}. \tag{4.1}$$

For simplicity we shall work with the corresponding functions

$$\varphi_n(q) = 2\mathrm{i}\psi_n(q)\mathrm{e}^{-\frac{1}{2}q^2}. \tag{4.2}$$

To begin our study of these functions we obtain recurrence relations.

Proposition 4.1.
(a)

$$q\varphi_n(q) = \sqrt{\frac{n+1}{2}}\varphi_{n+1}(q) + \sqrt{\frac{n}{2}}\varphi_{n-1}(q) + \frac{1+(-1)^n}{\sqrt{2(n+1)}}h_{n+1}(q) + \frac{1-(-1)^n}{\sqrt{2n}}h_{n-1}(q) \tag{4.3a}$$

(b)

$$\varphi_n'(q) = -\sqrt{\frac{n+1}{2}}\varphi_{n+1}(q) + \sqrt{\frac{n}{2}}\varphi_{n-1}(q) - \frac{1+(-1)^n}{\sqrt{2(n+1)}}h_{n+1}(q) + \frac{1-(-1)^n}{\sqrt{2n}}h_{n-1}(q) - 2\frac{1-(-1)^n}{q}h_n(q) \tag{4.3b}$$

for $n \geqslant 0$.

Proof. Consulting equation (2.2), a direct calculation shows that the commutator of Q and $\mathcal{Z}$ is given by

$$([Q,\mathcal{Z}]g)(q) = \mathrm{e}^{-\frac{1}{2}q^2}\int_{-q}^{q}\mathrm{e}^{\frac{1}{2}p^2}g(p)\,\mathrm{d}p + \mathrm{e}^{\frac{1}{2}q^2}\int_q^\infty \mathrm{e}^{-\frac{1}{2}p^2}\{g(p)-g(-p)\}\,\mathrm{d}p$$

for any $g \in \mathcal{S}(\mathbb{R})$ and $q > 0$. For $g = h_n$ these integrals can be done in closed form, and we find that

$$([Q,\mathcal{Z}]h_n)(q) = \frac{1+(-1)^n}{\sqrt{2(n+1)}}h_{n+1}(q) + \frac{1-(-1)^n}{\sqrt{2n}}h_{n-1}(q)$$

for $n \geqslant 0$ and $q > 0$. Applying the operator $\mathcal{Z}$ to the identity

$$\sqrt{2(n+1)}h_{n+1} - 2Qh_n + \sqrt{2n}h_{n-1} = 0$$

we obtain (a).

Again, direct calculation shows that

$$(\mathcal{Z}g)'(q) = \frac{2g(-q)}{q} - \mathrm{e}^{-\frac{1}{2}q^2}\int_{-q}^{q}\mathrm{e}^{\frac{1}{2}p^2}g(p)\,\mathrm{d}p + \mathrm{e}^{\frac{1}{2}q^2}\int_q^\infty \mathrm{e}^{-\frac{1}{2}p^2}\{g(p)-g(-p)\}\,\mathrm{d}p + (\mathcal{Z}g')(q)$$

for any $g \in \mathcal{S}(\mathbb{R})$ and $q > 0$. These integrals are the same as those above, so

$$(\mathcal{Z}h_n)'(q) = \frac{2(-1)^n h_n(q)}{q} - \frac{1+(-1)^n}{\sqrt{2(n+1)}}h_{n+1}(q) + \frac{1-(-1)^n}{\sqrt{2n}}h_{n-1}(q) + (\mathcal{Z}h_n')(q)$$

for $n \geqslant 0$ and $q > 0$. Now apply $\mathcal{Z}$ to the identity

$$\sqrt{2(n+1)}h_{n+1} + 2h_n' - \sqrt{2n}h_{n-1} = 0$$

yielding (b). □

Thus we see that the functions φ_n obey recurrence relations which are essentially inhomogenous versions of those satisfied by the Hermite functions h_n. This pattern persists, for if we introduce the standard creation and annihilation operators

$$A^+ = \frac{1}{\sqrt{2}}\left(q - \frac{\mathrm{d}}{\mathrm{d}q}\right) \qquad A = \frac{1}{\sqrt{2}}\left(q + \frac{\mathrm{d}}{\mathrm{d}q}\right) \tag{4.4a}$$

we can write

$$h_n = \frac{(A^+)^n}{\sqrt{n!}} h_0 \qquad A h_0 = 0 \tag{4.4b}$$

so that

$$A^+ h_n = \sqrt{n+1}\, h_{n+1} \qquad A h_n = \sqrt{n}\, h_{n-1} \tag{4.4c}$$

and so obtain

$$[A\varphi_n](q) = \sqrt{n}\varphi_{n-1} + \frac{1-(-1)^n}{\sqrt{n}} h_{n-1}(q) - \sqrt{2}\frac{1-(-1)^n}{q} h_n(q) \tag{4.5a}$$

$$[A^+\varphi_n](q) = \sqrt{n+1}\varphi_{n+1} + \frac{1+(-1)^n}{\sqrt{n+1}} h_{n+1}(q) + \sqrt{2}\frac{1-(-1)^n}{q} h_n(q) \tag{4.5b}$$

for all $n \geqslant 0$. Calculating the action of A^+A (the number operator) on φ_{2n}, we obtain the inhomogeneous differential equation

$$\varphi''_{2n}(q) + [4n+1-q^2]\varphi_{2n}(q) = -\frac{8\sqrt{n}}{q} h_{2n-1}(q) \tag{4.6}$$

for example.

5. Coherent states and the operators Δ ($e^{i\varphi}$) and Δ ($e^{-i\varphi}$)

Let us return now to considering the coherent state Φ_α for $\alpha \in \mathbb{C}$, and look at the first two moments of the operators $\Delta(e^{i\varphi})$ and $\Delta(e^{-i\varphi})$ when calculated in these states.

It should be noted that many of the quantities calculated in relation to laser phase experiments, e.g. see [20], seem to be calculating expectations and variances of quantities such as $\cos\varphi$ and $\sin\varphi$, and not φ directly. Thus it seems that the correct quantum mechanical approach might be to calculate the moments for operators such as $\Delta(\cos\varphi)$, $\Delta(\sin\varphi)$, $\Delta(e^{i\varphi})$ and $\Delta(e^{-i\varphi})$ and compare these with the experimental data. We shall have to wait for experiments that measure $\Delta(\varphi)$ directly. The operator $e^{i\Delta(\varphi)}$ could be added to the list, but we do not yet know enough about $\Delta(\varphi)$ to be able to do the calculations.

Interpreting the experimental results as the measurement of one operator rather than another is a delicate matter, particularly in view of the many possible candidates now known. The situation is further complicated if the results of Barnett, Pegg and others who use their theory is taken into account. That theory is usually presented as defined by the moments of a Hermitian phase operator, but on a Hilbert space not unitarily equivalent to $L^2(\mathbb{R})$. However, as we show in a subsequent paper [19], their theory can be written in terms of $L^2(\mathbb{R})$, but requiring a family of operators, so that, e.g., determining the uncertainty in a state requires two operators, as opposed to the one phase operator needed in any of the other models.

It should be noted that care needs to be taken with all of these calculations. For example, it may be necessary to consider an operator such as

$$\Delta(\cos\varphi)^2 + \Delta(\sin\varphi)^2.$$

However, since

$$\Delta(\cos\varphi)^2 \neq \Delta(\cos^2\varphi) \qquad \text{and} \qquad \Delta(\sin\varphi)^2 \neq \Delta(\sin^2\varphi)$$

it follows that

$$\Delta(\cos\varphi)^2 + \Delta(\sin\varphi)^2 \neq I.$$

Because of this, certain of our results contain terms without an analogue in previous works concerning operator forms for $\cos\varphi$ and $\sin\varphi$, although these additional terms do not affect first-order asymptotic behaviour.

Let us consider the coherent state

$$\Phi_\alpha(q) = \frac{1}{\pi^{1/4}} \exp[\tfrac{1}{4}(\bar{\alpha}^2 - |\alpha|^2)] \exp[-\tfrac{1}{2}q^2 + \mathrm{i}\bar{\alpha}q] \tag{5.1}$$

and set

$$\alpha = R\mathrm{e}^{\mathrm{i}\theta} \qquad R > 0,\ -\pi < \theta < \pi$$

the quantities R and θ will keep this fixed significance from now on.

Recall that the symbol $\mathcal{G}$ stands for the Wigner–Weyl transformation introduced at the end of the first section. We also recall the precise meaning of $\mathrm{e}^{\pm\mathrm{i}\varphi}$ as a function in phase space [2]:

$$\mathrm{e}^{\pm\mathrm{i}\varphi}(p,q) = \begin{cases} (p \pm \mathrm{i}q)/(p^2+q^2)^{1/2} & \text{if } p^2+q^2 > 0 \\ 1 & \text{if } p = q = 0. \end{cases} \tag{5.2}$$

In terms of $z = p + \mathrm{i}q$ and $r = |z|$ this can be written as

$$[\mathrm{e}^{\mathrm{i}\varphi}](z) = \begin{cases} z/r & \text{if } r > 0 \\ 1 & \text{if } r = 0 \end{cases} \tag{5.3a}$$

and

$$[\mathrm{e}^{-\mathrm{i}\varphi}](z) = \begin{cases} \bar{z}/r & \text{if } r > 0 \\ 1 & \text{if } r = 0. \end{cases} \tag{5.3b}$$

The brackets are meant to emphasize that (z) is an argument of the function and not multiplication by z. Note that

$$[\mathrm{e}^{\pm\mathrm{i}\varphi}](\mathrm{e}^{\mathrm{i}\beta}z) = \mathrm{e}^{\pm\mathrm{i}\beta}[\mathrm{e}^{\pm\mathrm{i}\varphi}](z). \tag{5.4}$$

Then

$$[\mathcal{G}(\overline{\Phi_\alpha} \otimes \Phi_\alpha)](p,q) = \frac{1}{\pi}\mathrm{e}^{-|z-\alpha|^2} \qquad z = p + \mathrm{i}q \tag{5.5}$$

which is easily verified. Shifting arguments $z \to z + \alpha$ and then $z \to \mathrm{e}^{-\mathrm{i}\theta}z$, we find that

$$\begin{aligned} \langle \Phi_\alpha, \Delta(\mathrm{e}^{\pm\mathrm{i}\varphi})\Phi_\alpha \rangle &= \frac{1}{\pi}\int_C [\mathrm{e}^{\pm\mathrm{i}\varphi}](z+\alpha)\mathrm{e}^{-|z|^2}\,\mathrm{d}A(z) \\ &= \frac{1}{\pi}\int_C [\mathrm{e}^{\pm\mathrm{i}\varphi}](\mathrm{e}^{\mathrm{i}\theta}z + \mathrm{e}^{\mathrm{i}\theta}R)\mathrm{e}^{-|z|^2}\,\mathrm{d}A(z) \\ &= \frac{\mathrm{e}^{\pm\mathrm{i}\theta}}{\pi}\int_C [\mathrm{e}^{\pm\mathrm{i}\varphi}](z+R)\mathrm{e}^{-|z|^2}\,\mathrm{d}A(z) \\ &= \mathrm{e}^{\pm\mathrm{i}\theta}\langle \Phi_R, \Delta(\mathrm{e}^{\pm\mathrm{i}\varphi})\Phi_R \rangle. \end{aligned} \tag{5.6}$$

Here and subsequently, we write $\mathrm{d}A(z)$ for the area element in phase space, written in terms of complex coordinates. Note that these are not line integrals in the sense of Cauchy.

We can evaluate this in terms of modified Bessel functions of the first kind:

$$\begin{aligned} \langle \Phi_R, \Delta(\mathrm{e}^{\pm\mathrm{i}\varphi})\Phi_R \rangle &= \frac{1}{\pi}\int_0^\infty \int_{-\pi}^{\pi} \mathrm{e}^{\pm\mathrm{i}\xi}\mathrm{e}^{-r^2+2rR\cos\xi-R^2} r\,\mathrm{d}\xi\,\mathrm{d}r \\ &= \frac{2}{\pi}\mathrm{e}^{-R^2}\int_0^\infty \int_0^\pi \cos\xi\,\mathrm{e}^{-r^2+2rR\cos\xi} r\,\mathrm{d}\xi\,\mathrm{d}r \end{aligned}$$

$$= e^{-R^2} \int_R e^{-x^2} I_1(2Rx) x \, dx$$
$$= \frac{\pi^{1/2}}{2} R e^{-\frac{1}{2}R^2} [I_0(\tfrac{1}{2}R^2) + I_1(\tfrac{1}{2}R^2)]. \tag{5.7}$$

Note that this agrees with the calculations of Freyberger and Schleich [21], cf equation (11).

We are also able to determine the asymptotic form of this matrix element for large R:

$$\langle \Phi_R, \Delta(e^{\pm i\varphi})\Phi_R \rangle = \frac{2R}{\pi^{1/2}} \int_0^{\pi/2} \cos^2 \xi \, e^{-R^2 \sin^2 \xi} \, d\xi$$
$$= \frac{2R}{\pi^{1/2}} \int_0^1 \sqrt{1-u^2} e^{-R^2 u^2} \, du$$
$$= 1 - \frac{1}{4R^2} + O\left(\frac{1}{R^4}\right) \qquad R \to \infty. \tag{5.8}$$

Thus we can write the expectation of $\Delta(e^{\pm i\varphi})$ in the coherent state Φ_α:

$$\langle \Phi_\alpha, \Delta(e^{\pm i\varphi})\Phi_\alpha \rangle = \frac{\pi^{1/2}}{2} R e^{-\frac{1}{2}R^2} e^{\pm i\theta} [I_0(\tfrac{1}{2}R^2) + I_1(\tfrac{1}{2}R^2)] \tag{5.9}$$

and its asymptotic form:

$$\langle \Phi_\alpha, \Delta(e^{\pm i\varphi})\Phi_\alpha \rangle = \left[1 - \frac{1}{4R^2}\right] e^{\pm i\theta} + O\left(\frac{1}{R^4}\right) \qquad R \to \infty. \tag{5.10}$$

With these two formulae, we have obtained the first moments of $\Delta(e^{i\varphi})$ and $\Delta(e^{-i\varphi})$ in the coherent state Φ_α, and their leading asymptotic forms.

We now seek the asymptotic behaviour of the norm of $\Delta(e^{\pm i\varphi})$ acting on the state Φ_α, which we find via the Hermite functions. In [2] we determined that $\Delta(e^{\pm i\varphi})$ were shift operators when acting on the Hermite basis, with

$$\Delta(e^{i\varphi}) h_n = \lambda_{n+1} h_{n+1} \tag{5.11a}$$

and

$$\Delta(e^{-i\varphi}) h_n = \overline{\lambda_n} h_{n-1} \tag{5.11b}$$

where $\lambda_0 = 0$ and

$$\lambda_{n+1} = i\sqrt{\frac{n+1}{2}} \frac{\Gamma(\frac{1}{2}n + s_n)}{\Gamma(\frac{1}{2}n + \frac{1}{2} + s_n)} \tag{5.11c}$$

and where s_n is defined in equation (3.12). Using Stirling's formula we deduce that

$$\lambda_{n+1} = i\left[1 + \frac{(-1)^n}{4n} + O\left(\frac{1}{n^2}\right)\right] \qquad n \to \infty.$$

From the definition of $O(1/n^2)$, it follows that there exists a bounded sequence

$$\sup_n |a_n| < \infty$$

such that

$$|\lambda_n|^2 = 1 - \frac{(-1)^n}{2n} + \frac{a_n}{n^2} \qquad n \geqslant 1.$$

Using equation (2.6) to express Φ_α in terms of Hermite functions, we have

$$\begin{aligned}\|\Delta(e^{i\varphi})\Phi_\alpha\|^2 &= \|\Delta(e^{i\varphi})\Phi_R\|^2 \\ &= e^{-\frac{1}{2}R^2}\sum_{n\geqslant 0}\frac{R^{2n}}{2^n n!}|\lambda_{n+1}|^2 \\ &= 1+\frac{1}{R^2}e^{-\frac{1}{2}R^2}(1-e^{-\frac{1}{2}R^2})+e^{-\frac{1}{2}R^2}\sum_{n\geqslant 0}\frac{R^{2n}}{2^n}\frac{a_{n+1}}{(n+1)(n+1)!}.\end{aligned}$$

Hence there exists a positive constant A such that

$$|\,\|\Delta(e^{i\varphi})\Phi_R\|^2-1| \leqslant \frac{1}{R^2}e^{-\frac{1}{2}R^2}+\frac{A}{R^4}.$$

Thus

$$\|\Delta(e^{i\varphi})\Phi_\alpha\|^2 = \|\Delta(e^{i\varphi})\Phi_R\|^2 = 1+O\left(\frac{1}{R^4}\right) \qquad R\to\infty. \tag{5.12}$$

Does a coherent state approximate an eigenvector of the phase in some sense? To answer this question we consider the asymptotic form of

$$\begin{aligned}\|[\Delta(e^{i\varphi})-e^{i\theta}]\Phi_\alpha\|^2 &= \|\Delta(e^{i\varphi})\Phi_\alpha\|^2-2\,\mathrm{Re}[e^{-i\theta}\langle\Phi_\alpha,\Delta(e^{i\varphi})\Phi_\alpha\rangle+1 \\ &= \|\Delta(e^{i\varphi})\Phi_R\|^2-2\,\mathrm{Re}[\langle\Phi_R,\Delta(e^{i\varphi})\Phi_R\rangle]+1 \\ &= \|[\Delta(e^{i\varphi})-1]\Phi_R\|^2 \\ &= \frac{1}{2R^2}+O\left(\frac{1}{R^4}\right) \qquad R\to\infty.\end{aligned} \tag{5.13a}$$

Re stands for the real part of a complex number; we indicate the imaginary part by Im. A similar calculation for the opposite phase yields

$$\begin{aligned}\|[\Delta(e^{-i\varphi})-e^{-i\theta}]\Phi_\alpha\|^2 &= \|[\Delta(e^{-i\varphi})-1]\Phi_R\|2 \\ &= \frac{1}{2R^2}+O\left(\frac{1}{R^4}\right) \qquad R\to\infty.\end{aligned} \tag{5.13b}$$

Thus

$$\lim_{R\to\infty}\|[\Delta(e^{i\varphi})-e^{i\theta}]\Phi_{Re^{i\theta}}\|^2 = 0 \tag{5.14a}$$

and

$$\lim_{R\to\infty}\|[\Delta(e^{-i\varphi})-e^{-i\theta}]\Phi_{Re^{i\theta}}\|^2 = 0 \tag{5.14b}$$

which means that $e^{i\theta}$ is an approximate eigenvalue of $\Delta(e^{i\varphi})$ and $e^{-i\theta}$ is an approximate eigenvalue of $\Delta(e^{-i\varphi})$, with

$$\{\Phi_{Re^{i\theta}} : R>0\}$$

providing a common family of approximating unit vectors. However, we already know that $e^{i\theta}$ (respectively $e^{-i\theta}$) is not an eigenvalue of $\Delta(e^{i\varphi})$ (respectively $\Delta(e^{-i\varphi})$), so this observation is at best approximate [2].

By way of comparison, we note that Freyberg and Schleich provide an interesting insight into the results of some laser experiments relating to phase. Their analysis places considerable importance on a quantity they call the dispersion, which is, essentially, the sum of the variances of $\Delta(\cos\varphi)$ and $\Delta(\sin\varphi)$ in the state Φ_α (in our notation). Their calculations are semiclassical, however, as they impose the condition that, in effect,

$$\Delta(\cos\varphi)^2+\Delta(\sin\varphi)^2 = I.$$

As we have noted above, this is *not* strictly appropriate in the full quantum mechanical context.

With this in mind, let us calculate the asymptotic form of the analogous quantity,

$$\delta(\alpha) = \mathrm{var}[\Delta(\cos\varphi); \Phi_\alpha] + \mathrm{var}[\Delta(\sin\varphi); \Phi_\alpha] \tag{5.15a}$$

in the Wigner–Weyl picture. Evidently

$$\delta(\alpha) = \langle \Phi_\alpha, [\Delta(\cos\varphi)^2 + \Delta(\sin\varphi)^2]\Phi_\alpha\rangle - |\langle \Phi_\alpha, \Delta(\mathrm{e}^{\mathrm{i}\varphi})\Phi_\alpha\rangle|^2. \tag{5.15b}$$

Now, in detail,

$$\begin{aligned}\Delta(\cos\varphi)^2 + \Delta(\sin\varphi)^2 &= \frac{1}{2^2}[\Delta(\mathrm{e}^{\mathrm{i}\varphi}) + \Delta(\mathrm{e}^{-\mathrm{i}\varphi})]^2 + \frac{1}{(2\mathrm{i})^2}[\Delta(\mathrm{e}^{\mathrm{i}\varphi}) - \Delta(\mathrm{e}^{-\mathrm{i}\varphi})]^2 \\ &= \tfrac{1}{2}\{\Delta(\mathrm{e}^{\mathrm{i}\varphi}), \Delta(\mathrm{e}^{-\mathrm{i}\varphi})\}_+ \\ &= \tfrac{1}{2}\{\Delta(\mathrm{e}^{\mathrm{i}\varphi}), \Delta(\mathrm{e}^{\mathrm{i}\varphi})^*\}_+\end{aligned}$$

where the +-bracket indicates the anti-commutator.

Taking matrix elements,

$$\langle \Phi_\alpha, [\Delta(\cos\varphi)^2 + \Delta(\sin\varphi)^2]\Phi_\alpha\rangle = \tfrac{1}{2}(\|\Delta(\mathrm{e}^{\mathrm{i}\varphi})\Phi_\alpha\|^2 + \|\Delta(\mathrm{e}^{-\mathrm{i}\varphi})\Phi_\alpha\|^2) \tag{5.16}$$

from which we deduce that

$$\delta(R\mathrm{e}^{\mathrm{i}\theta}) = \delta(R) = \frac{1}{2R^2} + O\left(\frac{1}{R^4}\right) \qquad R \to \infty. \tag{5.17}$$

We may interpret this as saying that a fully quantum mechanical calulation (using the Weyl quantization of our proposed observables) gives the same first order asymptotic behaviour as do the calculations of Freyberger and Schleich [21], and which models the experimental results quite well.

One reason why some workers may have made essentially semiclassical approximations typified by the above results is that it has often been assumed that the Wigner–Weyl transform of Φ_α, in our terminology

$$\mathcal{G}[\overline{\Phi_\alpha} \otimes \Phi_\alpha](p, q)$$

provides a density function for large α against which phase space observables can be integrated to find their moments. Now it is of course true that

$$\int_{R^2} X(p, q)\mathcal{G}[\overline{\Phi_\alpha} \otimes \Phi_\alpha](p, q)\,\mathrm{d}p\,\mathrm{d}q = \langle \Phi_\alpha, \Delta(X)\Phi_\alpha\rangle \tag{5.18a}$$

for any observable X, but then

$$\int_{R^2} X(p, q)^2\mathcal{G}[\overline{\Phi_\alpha} \otimes \Phi_\alpha](p, q)\,\mathrm{d}p\,\mathrm{d}q = \langle \Phi_\alpha, \Delta(X^2)\Phi_\alpha\rangle \tag{5.18b}$$

which is not the same as

$$\int_{R^2} (X * X)(p, q)\mathcal{G}[\overline{\Phi_\alpha} \otimes \Phi_\alpha](p, q)\,\mathrm{d}p\,\mathrm{d}q = \langle \Phi_\alpha, \Delta(X)^2\Phi_\alpha\rangle \tag{5.18c}$$

in general. Using $\mathcal{G}[\overline{\Phi_\alpha} \otimes \Phi_\alpha]$ as a 'probability density' to calculate the variance of a phase space function X does not correspond exactly to calculating the variance of the quanum mechanical observable $\Delta(X)$ in the state Φ_α.

Of course it might be expected that $\Delta(X^2)$ and $\Delta(X)^2$ are very close to each other in certain circumstances, but when asymptotic behaviour is being investigated, such an assumption warrants extreme caution. Indeed, in the next section, we shall show how just this sort of assumption leads to inaccuracies.

Moreover, it should be noted that, in general, $\mathcal{G}[\bar{f} \otimes f](p, q)$ is not a positive function on phase space for $f \in \mathcal{S}(\mathbb{R})$. Indeed, this only happens for Gaussian functions f such as Φ_α (Hudson's theorem [22]). Thus, interpreting the Wigner–Weyl transform of Φ_α as a probability distribution even when $|\alpha|$ is large can be dangerous.

6. Coherent states and the operators $\Delta(\varphi)$

Moving on to consider the moments of the phase operator $\Delta(\varphi)$ with respect to coherent states, we begin by calculating that

$$\langle \Phi_\alpha, \Delta(\varphi)\Phi_\alpha \rangle = \frac{1}{\pi}\int_C \varphi(z) e^{-|z-\alpha|^2}\, dA(z) = \frac{1}{\pi}\int_C \varphi(e^{i\theta} z) e^{-|z-R|^2}\, dA(z). \tag{6.1}$$

For simplicity we shall work with the case $0 \leqslant \theta < \pi$. It is then clear that

$$\varphi(e^{i\theta} z) = \varphi(p, q) + \theta - 2\pi\, E_{\pi-\theta}(p, q) \tag{6.2a}$$

where the phase space function $E_{\pi-\theta}$ is given by the rule [2]

$$E_{\pi-\theta}(r\cos\beta, r\sin\beta) = \begin{cases} 1 & \text{if } r > 0 \text{ and } \pi - \theta < \beta < \pi \\ 0 & \text{otherwise.} \end{cases} \tag{6.2b}$$

In this notation it follows that

$$\langle \Phi_\alpha, \Delta(\varphi)\Phi_\alpha \rangle = \langle \Phi_R, \Delta(\varphi)\Phi_R \rangle + \theta - 2\pi \langle \Phi_R, \Delta(E_{\pi-\theta})\Phi_R \rangle.$$

Since

$$\varphi(p, -q) = -\varphi(p, q)$$

we have

$$\langle \Phi_R, \Delta(\varphi)\Phi_R \rangle = 0$$

and so

$$\begin{aligned} \langle \Phi_\alpha, \Delta(\varphi)\Phi_\alpha \rangle &= \theta - 2\pi \langle \Phi_R, \Delta(E_{\pi-\theta})\Phi_R \rangle \\ &= \theta - 2\pi \int_C E_{\pi-\theta}(z + R) e^{-|z|^2}\, dA(z). \end{aligned}$$

The geometry of the problem is slightly different according to whether θ is greater than or equal to $\pi/2$ or less than $\pi/2$, so let us define the function $k : [0, \pi) \to (0, 1]$ by setting

$$k(\theta) = \begin{cases} 1 & \text{if } 0 \leqslant \theta \leqslant \frac{\pi}{2} \\ \sin\theta & \text{if } \frac{\pi}{2} < \theta < \pi \end{cases}$$

so that

$$E_{\pi-\theta}(z + R) = 0 \qquad \text{for all } |z| < Rk(\theta).$$

Thus

$$|\langle \Phi_\alpha, \Delta(\varphi)\Phi_\alpha \rangle - \theta| \leqslant 2\pi \int_{|z| \geqslant Rk(\theta)} e^{-|z|^2}\, dA(z)$$

and so

$$|\langle \Phi_\alpha, \Delta(\varphi)\Phi_\alpha \rangle - \theta| \leqslant 2\pi^2 e^{-k(\theta)^2 R^2}. \tag{6.3}$$

The asymptotic behaviour of the expectation of $\Delta(\varphi)$ in the state Φ_α can be read off from this:

$$\langle \Phi_\alpha, \Delta(\varphi)\Phi_\alpha \rangle = \theta + O(e^{-k(\theta)^2 R^2}) \qquad R \to \infty. \tag{6.4}$$

In order to calculate the second moment of $\Delta(\varphi)$ in the state Φ_α, we need a more sophisticated approach. We have found it convenient to define the operators

$$[\mathcal{T}_a f](x) = f(x-a) \tag{6.5a}$$

$$[\mathcal{M}_a f](x) = e^{iax} f(x) \tag{6.5b}$$

for all $f \in \mathcal{S}(\mathbb{R})$ and $a \in \mathbb{R}$. We shall also employ the parity operator previously defined:

$$[\mathcal{R}f](x) = f(-x). \tag{6.5c}$$

With this notation,

$$[\mathcal{G}(\bar{f} \otimes g)](p,q) = \frac{1}{\pi} e^{2ipq} \langle \mathcal{M}_{2p}\mathcal{T}_{2q}\mathcal{R}f, g\rangle \qquad f, g \in \mathcal{S}(\mathbb{R}).$$

We can translate in phase space and write

$$[\mathcal{G}(\overline{\mathcal{M}_{2p}\mathcal{T}_{2q}\mathcal{R}f} \otimes g)](\xi,\eta) = e^{2i(\xi q - \eta p - pq)}[\mathcal{G}(\overline{\mathcal{R}f} \otimes g)](\xi - p, \eta - q).$$

The point of doing this is that for any phase space functions X, Y which are sufficiently regular so that $\Delta(X)$, $\Delta(Y)$ are continuous linear operators from $\mathcal{S}(\mathbb{R})$ to $L^2(\mathbb{R})$, we can write

$$[\mathcal{G}(\bar{f} \otimes \Delta(Y)g)](p,q) = \frac{1}{\pi}\int_\Pi Y(\xi,\eta) e^{2i(\xi q - \eta p)}[\mathcal{G}(\overline{\mathcal{R}f} \otimes g)](\xi - p, \eta - q)\, d\xi\, d\eta \tag{6.6a}$$

and so

$$\langle \Delta(X)f, \Delta(Y)g\rangle = \frac{1}{\pi}\int_\Pi \overline{X(p,q)} \times\left(\int_\Pi Y(\xi,\eta) e^{2i(\xi q - \eta p)}[\mathcal{G}(\overline{\mathcal{R}f} \otimes g)](\xi - p, \eta - q)\, d\xi\, d\eta\right) dp\, dq \tag{6.6b}$$

for any f, $g \in \mathcal{S}(\mathbb{R})$. We note that, viewed as a function on $\Pi \times \Pi$, that is, $\mathbb{R}^4$, the integrand may not be Lebesgue integrable. However, the integral exists as an iterated Lebesgue integral. This means that everything is well defined and we may proceed, being careful not to invoke Fubini's theorem as a matter of course.

Our first step is to consider the integrand of the inner integral with f and g equal to the coherent state Φ_α: with

$$[\mathcal{G}(\overline{\mathcal{R}\Phi_\alpha} \otimes \Phi_\alpha)](p,q) = \frac{1}{\pi} e^{-p^2 - q^2 - 2ip\,\mathrm{Im}\,\alpha + 2iq\,\mathrm{Re}\,\alpha}$$

we obtain

$$\frac{1}{\pi} e^{2i(\xi q - \eta p)}[\mathcal{G}(\overline{\mathcal{R}\Phi_\alpha} \otimes \Phi_\alpha)](\xi - p, \eta - q) = \frac{1}{\pi^2} e^{-|z-\alpha|^2 - |w-\alpha|^2 + 2(z-\alpha)\overline{(w-\alpha)}}$$

where in addition to setting $z = p + iq$ we also set $w = \xi + i\eta$.

In this notation,

$$\begin{aligned}
\|\Delta(X)\Phi_\alpha\|^2 &= \frac{1}{\pi^2}\int_C \overline{X(z+\alpha)}\left(\int_C X(w+\alpha) e^{-|z|^2 - |w|^2 + 2z\bar{w}}\, dA(w)\right) dA(z) \\
&= \frac{1}{\pi^2}\int_C \overline{X[e^{i\theta}(z+R)]}\left(\int_C X[e^{i\varphi}(w+R)] e^{-|z|^2 - |w|^2 + 2z\bar{w}}\, dA(w)\right) dA(z) \\
&= \|\Delta(X_{[\theta]})\Phi_R\|^2
\end{aligned}$$

where

$$X_{[\theta]}(p,q) = X(e^{i\theta}z) \qquad z = p + iq. \tag{6.7}$$

As we did before, we restrict the angle to the interval $0 \leqslant \theta < \pi$. Then this last result applied to $\Delta(\varphi)$ leads us to the calculation

$$\begin{aligned}\|\Delta(\varphi)\Phi_\alpha\|^2 &= \|[\Delta(\varphi) + \theta - 2\pi\Delta(E_{\pi-\theta})]\Phi_R\|^2 \\ &= \|\Delta(\varphi)\Phi_R\|^2 + \theta^2 + 4\pi^2\|\Delta(E_{\pi-\theta})\Phi_R\|^2 + 2\theta\langle\Phi_R, \Delta(\varphi)\Phi_R\rangle \\ &\quad -4\pi\theta\langle\Phi_R, \Delta(E_{\pi-\theta})\Phi_R\rangle - 4\pi\,\mathrm{Re}\langle\Delta(\varphi)\Phi_R, \Delta(E_{\pi-\theta})\Phi_R\rangle \\ &= \|\Delta(\varphi)\Phi_R\|^2 + \theta^2 + 4\pi^2\|\Delta(E_{\pi-\theta})\Phi_R\|^2 \\ &\quad -4\pi\theta\langle\Phi_R, \Delta(E_{\pi-\theta})\Phi_R\rangle - 4\pi\,\mathrm{Re}\langle\Delta(\varphi)\Phi_R, \Delta(E_{\pi-\theta})\Phi_R\rangle. \end{aligned} \tag{6.8}$$

We have already shown that

$$\langle\Phi_R, \Delta(E_{\pi-\theta})\Phi_R\rangle = O(\mathrm{e}^{-k(\theta)^2R^2}) \qquad R \to \infty. \tag{6.9}$$

Therefore, to determine the asymptotic behaviour of $\|\Delta(\varphi)\Phi_\alpha\|^2$ we must discover the asymptotic form of $\|\Delta(\varphi)\Phi_R\|$ and $\|\Delta(E_{\pi-\theta})\Phi_R\|$. An asymptotic bound on the cross term $\langle\Delta(\varphi)\Phi_R, \Delta(E_{\pi-\theta})\Phi_R\rangle$ will then follow from the Cauchy–Schwarz inequality.

For every integer $n \geqslant 0$ we define the function

$$\begin{aligned} U_n(R,\theta) &= \int_C E_{\pi-\theta}(z+R)\mathrm{e}^{-|z|^2}z^n\,\mathrm{d}A(z) \\ &= \int_{|z|\geqslant k(\theta)R} E_{\pi-\theta}(z+R)\mathrm{e}^{-|z|^2}z^n\,\mathrm{d}A(z). \end{aligned} \tag{6.10}$$

We recognize the expression when $n = 0$ as

$$U_0(R,\theta) = \langle\Phi_R, \Delta(E_{\pi-\theta})\Phi_R\rangle \tag{6.11}$$

whose asymptotic form we know. To find the asymptotic form of $\|\Delta(E_{\pi-\theta})\Phi_R\|$ we shall have to consider a weighted sum of the U_n over n, so we need estimates of $|U_n|$.

For any $r > k(\theta)R$ we can find functions μ and ν satisfying

$$0 \leqslant \mu(r,\theta) < \nu(r,\theta) < \pi$$

such that

$$E_{\pi-\theta}(r\cos\beta + R, r\sin\beta) = \begin{cases} 1 & \text{if } \pi - \nu(r,\theta) < \beta < \pi - \mu(r,\theta) \\ 0 & \text{otherwise.} \end{cases} \tag{6.12}$$

Then

$$\begin{aligned} |U_n(R,\theta)| &= \left| \int_{k(\theta)R}^{\infty} \int_{\pi-\nu(r,\theta)}^{\pi-\mu(r,\theta)} \mathrm{e}^{-r^2} r^{n+1} \mathrm{e}^{\mathrm{i}n\beta}\,\mathrm{d}\beta\,\mathrm{d}r \right| \\ &= \left| \frac{(-1)^n}{\mathrm{i}n} \int_{k(\theta)R}^{\infty} \mathrm{e}^{-r^2} r^{n+1} [\mathrm{e}^{-\mathrm{i}n\mu(r,\theta)} - \mathrm{e}^{-\mathrm{i}n\nu(r,\theta)}]\,\mathrm{d}r \right| \\ &\leqslant \frac{2}{n} \int_{k(\theta)R}^{\infty} \mathrm{e}^{-r^2} r^{n+1}\,\mathrm{d}r \\ &= \frac{k(\theta)^n R^n}{n} \mathrm{e}^{-k(\theta)^2R^2} + \int_{k(\theta)R}^{\infty} \mathrm{e}^{-r^2} r^{n-1}\,\mathrm{d}r \end{aligned}$$

using integration by parts.

The sum we need is

$$\begin{aligned} \sum_{n\geqslant 1} \frac{2^n}{n!}|U_n(R,\theta)|^2 &\leqslant \sum_{n\geqslant 1} \frac{2^n}{n!} \left(\frac{k(\theta)^n R^n}{n} \mathrm{e}^{-k(\theta)^2R^2} + \int_{k(\theta)R}^{\infty} \mathrm{e}^{-r^2} r^{n-1}\,\mathrm{d}r \right)^2 \\ &\leqslant 2\sum_{n\geqslant 1} \frac{2^n}{n!} \left(\frac{k(\theta)^{2n} R^{2n}}{n^2} \mathrm{e}^{-2k(\theta)^2R^2} + \left[\int_{k(\theta)R}^{\infty} \mathrm{e}^{-r^2} r^{n-1}\,\mathrm{d}r \right]^2 \right). \end{aligned}$$

Using Fubini's theorem we may transform the integral as follows:

$$
\begin{aligned}
2\sum_{n\geqslant 1}\frac{2^n}{n!}\left[\int_{k(\theta)R}^{\infty} e^{-r^2} r^{n-1}\,dr\right]^2 &= 2\sum_{n\geqslant 1}\frac{2^n}{n!}\int_{k(\theta)R}^{\infty}\int_{k(\theta)R}^{\infty} e^{-x^2-y^2} x^{n-1} y^{n-1}\,dx\,dy \\
&= 2\int_{k(\theta)R}^{\infty}\int_{k(\theta)R}^{\infty}\frac{1}{xy} e^{-x^2-y^2}(e^{2xy}-1)\,dx\,dy \\
&\leqslant 2\int_{k(\theta)R}^{\infty}\int_{k(\theta)R}^{\infty}\frac{1}{xy} e^{-(x-y)^2}\,dx\,dy \\
&= 4\int_0^{\infty}\frac{1}{t}\log(1+t) e^{-k(\theta)^2R^2t^2}\,dt \\
&\leqslant 4\int_0^{\infty} e^{-k(\theta)^2R^2t^2}\,dt \\
&= \frac{2\pi^{1/2}}{k(\theta)R}.
\end{aligned}
$$

The first term may be estimated as well:

$$
\begin{aligned}
2\sum_{n\geqslant 1}\frac{2^n}{n!}\frac{k(\theta)^{2n}R^{2n}}{n^2} e^{-2k(\theta)^2R^2} &\leqslant 4\sum_{n\geqslant 1}\frac{2^n k(\theta)^{2n}R^{2n}}{(n+1)!} e^{-2k(\theta)^2R^2} \\
&\leqslant \frac{2}{k(\theta)^2R^2}.
\end{aligned}
$$

Combining these,

$$
\sum_{n\geqslant 1}\frac{2^n}{n!}|U_n(R,\theta)| \leqslant \frac{2}{k(\theta)^2R^2} + \frac{2\pi^{1/2}}{k(\theta)R} \tag{6.13}
$$

from which we deduce that the series converges and that it has the asymptotic expression

$$
\sum_{n\geqslant 1}\frac{2^n}{n!}|U_n(R,\theta)| = O\left(\frac{1}{R}\right) \qquad R\to\infty. \tag{6.14}
$$

The series we have just examined comes from the power series expansion of $e^{2z\bar{w}}$ in

$$
\int_C E_{\pi-\theta}(w+R) e^{-|w|^2+2z\bar{w}}\,dA(w) = \sum_{n\geqslant 0}\frac{2^n z^n}{n!}\overline{U_n(R,\theta)}. \tag{6.15a}
$$

Convergence is assured and we may integrate both sides and interchange the order of sum and integral to get:

$$
\begin{aligned}
\int_C E_{\pi-\theta}(z+R) e^{-|z|^2}&\left(\int_C E_{\pi-\theta}(w+R) e^{-|w|^2+2z\bar{w}}\,dA(w)\right) dA(z) \\
&= \int_C E_{\pi-\theta}(z+R) e^{-|z|^2}\left(\sum_{n\geqslant 0}\frac{2^n z^n}{n!}\overline{U_n(R,\theta)}\right) dA(z) \\
&= \sum_{n\geqslant 0}\frac{2^n}{n!}|U_n(R,\theta)|^2.
\end{aligned} \tag{6.15b}
$$

Thus it follows that

$$
\|\Delta(E_{\pi-\theta})\Phi_R\|^2 = O\left(\frac{1}{R}\right) \qquad R\to\infty. \tag{6.16}
$$

The approximations we have made are by no means the sharpest. It may in fact be true, for example, that

$$\|\Delta(E_{\pi-\theta})\Phi_R\|^2 = o\left(\frac{1}{R}\right) \qquad R \to \infty.$$

However, more detailed calculations below will give a more accurate assessment of the asymptotic behaviour of $\|\Delta(\varphi)\Phi_R\|^2$, and so it seems that to first order, a sharper result for $\|\Delta(E_{\pi-\theta})\Phi_R\|^2$ is probably not necessary.

It is to the asymptotic behaviour of $\|\Delta(\varphi)\Phi_R\|^2$ that we now turn. We begin by evaluating a singular integral. Let $\log(1+z)$ be the branch of the logarithm defined on $\mathbb{C} \setminus (-\infty, -1]$. Define

$$I_n(r) = \mathrm{Pv} \int_{|z|=r} z^{n-1} \log(1+z)\,\mathrm{d}z \qquad r > 0,\ n \in \mathbb{Z}. \tag{6.17}$$

Standard complex integration methods yield

$$I_n(r) = \begin{cases} 0 & \text{if } n \geqslant 0 \text{ and } 0 < r < 1 \\ \dfrac{2\pi\mathrm{i}(-1)^n}{n} & \text{if } n \leqslant -1 \text{ and } 0 < r < 1 \\ \dfrac{2\pi\mathrm{i}(-1)^n}{n}(r^n - 1) & \text{if } n \geqslant 1 \text{ and } r > 1 \\ 2\pi\mathrm{i}\log r & \text{if } n = 0 \text{ and } r > 1 \\ \dfrac{2\pi\mathrm{i}(-1)^n}{n} r^n & \text{if } n \leqslant -1 \text{ and } r > 1. \end{cases}$$

We must now consider integrals formed as U_n is, but with $\log(1+z)$ replacing $E_{\pi-\theta}$. We need two sequences of functions, to deal with complex conjugation. Thus we define

$$B_n(R) = \int_C \log(1+z)\mathrm{e}^{-R^2|z|^2} z^n\,\mathrm{d}A(z) \tag{6.18a}$$

and

$$C_n(R) = \int_C \log(1+z)\mathrm{e}^{-R^2|z|^2} \bar{z}^n\,\mathrm{d}A(z) \tag{6.18b}$$

for integers $n \geqslant 0$.

Utilizing the properties of the logarithm,

$$B_n(R) = \frac{1}{\mathrm{i}} \int_0^\infty r I_n(r)\mathrm{e}^{-R^2r^2}\,\mathrm{d}r \tag{6.19a}$$

and

$$C_n(R) = \frac{1}{\mathrm{i}} \int_0^\infty r^{2n+1} I_{-n}(r)\mathrm{e}^{-R^2r^2}\,\mathrm{d}r. \tag{6.19b}$$

Then

$$B_n(R) = \begin{cases} 2\pi \displaystyle\int_1^\infty r \log r \mathrm{e}^{-R^2r^2}\,\mathrm{d}r & \text{if } n = 0 \\ \dfrac{\pi(-1)^n}{2} R^{-(n+2)}\Gamma(\tfrac{1}{2}n, R^2) & \text{if } n \geqslant 1 \end{cases} \tag{6.20a}$$

and

$$C_n(R) = \begin{cases} B_0(R) & \text{if } n = 0 \\ \pi(-1)^{n-1} R^{-(2n+2)}\gamma(n, R^2) - B_n(R) & \text{if } n \geqslant 1 \end{cases} \tag{6.20b}$$

where

$$\Gamma(a,x)=\int_x^\infty e^{-t}t^{a-1}\,dt \qquad \text{and} \qquad \gamma(a,x)=\int_0^x e^{-t}t^{a-1}\,dt$$

are the incomplete gamma functions of Legendre; a is restricted by $\operatorname{Re} a>0$ in $\gamma(a,x)$.

These two sequences combine to make up the sequence $\{V_n\}$ which plays the same role for $\Delta(\varphi)$ that $\{U_n\}$ did for $E_{\pi-\theta}$:

$$V_n(R)=\int_C \varphi(z+1)e^{-R^2|z|^2}z^n\,dA(z)=\frac{1}{2i}[B_n(R)-\overline{C_n(R)}] \tag{6.21a}$$

and so

$$V_n(R)=\begin{cases}0 & \text{if } n=0\\ \dfrac{\pi(-1)^n}{2i}[\gamma(n,R^2)R^{-(2n+2)}+\Gamma(\frac{1}{2}n,R^2)R^{-(n+2)}] & \text{if } n\geqslant 1.\end{cases} \tag{6.21b}$$

As with U_n we shall need a certain infinite series in the V_n, and we now consider upper and lower bounds for that part which comes from the incomplete gamma function. Then, using Fubini's theorem as necessary,

$$\begin{aligned}\frac{1}{4}\sum_{n\geqslant 1}\frac{2^n}{n!}R^{-2n}\gamma(n,R^2)^2&=\frac{1}{4}\sum_{n\geqslant 1}\frac{2^n}{n!}R^{-2n}\int_0^{R^2}\int_0^{R^2}(st)^{n-1}e^{-s-t}\,ds\,dt\\&=\frac{1}{4}\int_0^{R^2}\int_0^{R^2}(st)^{-1}(e^{2st/R^2}-1)e^{-s-t}\,ds\,dt\\&\leqslant\frac{1}{2R^2}\int_0^{R^2}\int_0^{R^2}e^{2st/R^2-s-t}\,ds\,dt\\&=\frac{1}{R^2}\iint_{\substack{0\leqslant s,t\leqslant R^2\\ s+t\leqslant R^2}}e^{2st/R^2-s-t}\,ds\,dt.\end{aligned}$$

Having ended up with a planar integral, we can undo it into a more convenient form:

$$\begin{aligned}\frac{1}{4}\sum_{n\geqslant 1}\frac{2^n}{n!}R^{-2n}\gamma(n,R^2)^2&\leqslant\frac{1}{R^2}\int_0^{R^2}\left(\int_0^x e^{2t(x-t)/R^2-x}\,dt\right)dx\\&\leqslant\frac{1}{R^2}\int_0^{R^2}xe^{x^2/2R^2-x}\,dx\\&\leqslant\frac{1}{R^2}\int_0^{R^2}xe^{-\frac{1}{2}x}\,dx\\&\leqslant\frac{4}{R^2}.\end{aligned}$$

This is our upper bound. To find the lower bound we proceed as follows.

$$\begin{aligned}\frac{1}{4}\sum_{n\geqslant 1}\frac{2^n}{n!}R^{-2n}\gamma(n,R^2)^2&\geqslant\frac{1}{4}\iint_{\substack{0\leqslant s,t\leqslant R^2\\ s+t\leqslant R^2}}(st)^{-1}(e^{2st/R^2}-1)e^{-s-t}\,ds\,dt\\&=\frac{1}{4}\int_0^{R^2}\left(\int_0^x[t(x-t)]^{-1}(e^{2t(x-t)/R^2}-1)\,dt\right)e^{-x}\,dx\\&=\frac{1}{4}\int_0^{R^2}\left(\int_0^1[y(1-y)]^{-1}(e^{2x^2y(1-y)/R^2}-1)\,dy\right)x^{-1}e^{-x}\,dx\end{aligned}$$

$$\geqslant \frac{1}{2R^2}\int_0^{R^2} x\mathrm{e}^{-x}\,\mathrm{d}x$$
$$= \frac{1}{2R^2}[1-(R^2+1)\mathrm{e}^{-R^2}]$$
$$\geqslant \frac{1}{4R^2}$$

if $R > 2$.

Combining the two estimates,

$$\frac{1}{4R^2} \leqslant \frac{1}{4}\sum_{n\geqslant 1}\frac{2^n}{n!}R^{-2n}\gamma(n, R^2)^2 \leqslant \frac{4}{R^2} \tag{6.22}$$

if $R > 2$, and so certainly

$$\frac{1}{4}\sum_{n\geqslant 1}\frac{2^n}{n!}R^{-2n}\gamma(n, R^2)^2 = O\left(\frac{1}{R^2}\right) \qquad R\to\infty. \tag{6.23}$$

The series in which the other incomplete gamma function appears is treated similarly

$$\begin{aligned}\frac{1}{4}\sum_{n\geqslant 1}\frac{2^n}{n!}\Gamma(\tfrac{1}{2}n, R^2)^2 &= \frac{1}{4}\sum_{n\geqslant 1}\frac{2^n}{n!}\int_{R^2}^{\infty}\int_{R^2}^{\infty}(st)^{\frac{1}{2}n-1}\mathrm{e}^{-s-t}\,\mathrm{d}s\,\mathrm{d}t\\ &= \frac{1}{4}\int_{R^2}^{\infty}\int_{R^2}^{\infty}(st)^{-1}(\mathrm{e}^{2\sqrt{st}}-1)\mathrm{e}^{-s-t}\,\mathrm{d}s\,\mathrm{d}t\\ &= \int_R^{\infty}\int_R^{\infty}(xy)^{-1}(\mathrm{e}^{2xy}-1)\mathrm{e}^{-x^2-y^2}\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_R^{\infty}\int_R^{\infty}(xy)^{-1}\mathrm{e}^{-(x-y)^2}\,\mathrm{d}x\,\mathrm{d}y - \left(\int_R^{\infty}x^{-1}\mathrm{e}^{-x^2}\,\mathrm{d}x\right)^2\end{aligned}$$

With one more change of variable,

$$\tfrac{1}{4}\sum_{n\geqslant 1}\frac{2^n}{n!}\Gamma(\tfrac{1}{2}n, R^2)^2 = 2\int_0^{\infty}t^{-1}\log(1+t)\mathrm{e}^{-R^2t^2}\,\mathrm{d}t - \tfrac{1}{4}E_1(R^2)^2 \tag{6.24}$$

and so

$$\tfrac{1}{4}\sum_{n\geqslant 1}\frac{2^n}{n!}\Gamma(\tfrac{1}{2}n, R^2)^2 \leqslant 2\int_0^{\infty}\mathrm{e}^{-R^2t^2}\,\mathrm{d}t = \frac{\pi^{1/2}}{R}. \tag{6.25}$$

By E_1 we mean the function

$$E_1(x) = \int_x^{\infty}t^{-1}\mathrm{e}^{-t}\,\mathrm{d}t \qquad x>0$$

related to the exponential integral.

We conclude two things from this inequality. First, the series

$$\frac{R^4}{\pi^2}\sum_{n\geqslant 0}\frac{(2R^2)^n}{n!}|V_n(R)| < \infty \tag{6.26}$$

converges, and second, by using the method of Laplace, one of the standard techniques in the theory of asymptotic expansions, we find that

$$\frac{R^4}{\pi^2}\sum_{n\geqslant 0}\frac{(2R^2)^n}{n!}|V_n(R)| \sim \frac{\pi^{1/2}}{R} \qquad R\to\infty. \tag{6.27}$$

We are using the standard notation for asymptotic equivalence here.

As we did for U_n, we use a power series expansion to obtain

$$\begin{aligned}\int_C \varphi(w+R)\mathrm{e}^{-|w|^2+2z\bar{w}}\,\mathrm{d}A(w) &= \sum_{n\geqslant 0}\frac{2^n z^n}{n!}\int_C \varphi(w+R)\mathrm{e}^{-|w|^2}\bar{w}^n\,\mathrm{d}A(w)\\ &= \sum_{n\geqslant 0}\frac{2^n z^n R^{n+2}}{n!}\int_C \varphi(Rw+R)\mathrm{e}^{-R^2|w|^2}\bar{w}^n\,\mathrm{d}A(w)\\ &= \sum_{n\geqslant 0}\frac{2^n z^n R^{n+2}}{n!}\overline{V_n(R)}.\end{aligned}$$

Hence

$$\begin{aligned}\|\Delta(\varphi)\Phi_R\| &= \frac{1}{\pi^2}\int_C \varphi(z+R)\mathrm{e}^{-|z|^2}\left(\int_C \varphi(w+R)\mathrm{e}^{-|w|^2+2z\bar{w}}\,\mathrm{d}A(w)\right)\mathrm{d}A(z)\\ &= \frac{R^4}{\pi^2}\sum_{n\geqslant 0}\frac{2^n R^{2n}}{n!}|V_n(R)|.\end{aligned}$$

As before, the calculation is justified by the convergence of the final series. From this it follows that the asymptotic equivalence of the matrix element in question is

$$\|\Delta(\varphi)\Phi_R\| \sim \frac{\pi^{1/2}}{R} \qquad R\to\infty. \tag{6.28}$$

Combining the two parts of the calculation, we now see that

$$\|\Delta(\varphi)\Phi_\alpha\| = \theta^2 + O\left(\frac{1}{R}\right) \qquad R\to\infty \tag{6.29}$$

and so

$$\|[\Delta(\varphi)-\theta]\Phi_\alpha\| = O\left(\frac{1}{R}\right) \qquad R\to\infty. \tag{6.30}$$

Looking back at our assumptions, we have shown this only for $0\leqslant\theta<\pi$. However, the reflected case $-\pi<\theta\leqslant 0$ can be done in entirely analogous fashion. Doing so—we omit the calculation as being as long as the one we have presented, and of no independent interest—we learn that θ is an approximate eigenvalue of $\Delta(\varphi)$ for any $-\pi<\theta<\pi$, with $\{\Phi_{Re^{i\theta}} : R>0$ an approximating sequence of unit vectors. This implies that

$$[-\pi,\pi]\subseteq \mathrm{spec}[\Delta(\varphi)]. \tag{6.31}$$

By elementary spectral theory, this implies that

$$\pi\leqslant\|\Delta(\varphi)\|.$$

We already know from [2] that $\|\Delta(\varphi)\|\leqslant 3\pi/2$, so we can now say that

$$\pi\leqslant\|\Delta(\varphi)\|\leqslant 3\pi/2. \tag{6.32}$$

This sharpens considerably the results we had in [2], and increases our belief in the conjecture we stated there: that the spectrum of $\Delta(\varphi)$ is the continuous interval $[-\pi,\pi]$. A proof that the norm of $\Delta(\varphi)$ was equal to π would now prove this conjecture.

The particular result that $\|\Delta(\varphi)\Phi_R\|^2$ is asymptotically equivalent to $\pi^{1/2}/R$ is somewhat surprising, in view of what is expected from the literature. Conventional wisdom leads us to expect that the variance of $\Delta(\varphi)$ in the state Φ_R should behave like $(2R^2)^{-1}$ for large R, and we see this is not the case. However, the heuristic justification for the expected result relies on treating the Wigner–Weyl transform

$$W_R(p,q) = [\mathcal{G}(\overline{\Phi_R}\otimes\Phi_R)](p,q) = \frac{1}{\pi}\mathrm{e}^{-(p-R)^2-q^2} \tag{6.33}$$

as a probability density function in the phase plane, and then calculating the variance of φ classically using this distribution. However, as has been mentioned before, this is not correct. Certainly

$$\langle \Phi_R, \Delta(\varphi)\Phi_R \rangle = \int_\Pi \varphi(p,q) W_R(p,q)\,\mathrm{d}p\,\mathrm{d}q = 0 \tag{6.34}$$

but

$$\langle \Phi_R, \Delta(\varphi)^2\Phi_R \rangle \neq \int_\Pi \varphi(p,q)^2 W_R(p,q)\,\mathrm{d}p\,\mathrm{d}q. \tag{6.35}$$

All one can say is that

$$\langle \Phi_R, \Delta(\varphi)^2\Phi_R \rangle = \int_\Pi [\varphi * \varphi](p,q) W_R(p,q)\,\mathrm{d}p\,\mathrm{d}q \tag{6.36}$$

where $\varphi * \varphi$ is the Moyal product of φ with itself.

While it is tempting to say that $\varphi * \varphi$ is close to φ^2, and so use φ^2 to give approximate results, there are two problems connected with this.

Firstly, the expansion formulae which show the Moyal product $X * Y$ of two phase space functions as approximated by the pointwise product XY require that X and Y have good differentiability properties, which φ certainly does not possess: we do not have any useful knowledge of the relation between $\varphi * \varphi$ and φ^2.

Secondly, even if we knew that we could approximate $\varphi * \varphi$ by φ^2, the error in this approximation, even if small, may still have a significant effect in the asymptotic calculations which would ensue.

Moreover, as has been shown above, much of the evidence supporting the $1/2R^2$ asymptotic behaviour seems to be more properly directed towards the calculation of the variances of $\Delta(\mathrm{e}^{\mathrm{i}\varphi})$ and $\Delta(\mathrm{e}^{-\mathrm{i}\varphi})$ in coherent states; and in these cases our formalism *is* consistent with this reasoning and with the experimental evidence. It can be argued that, to date, experiments have not been performed which directly measure the angle φ, but rather measure the operators corresponding to $\cos\varphi$ and $\sin\varphi$. Evidently more remains to be done, both in calculating properties of $\Delta(\varphi)$ and in designing experiments to measure the angle directly. It might be, of course, that such experiments are not possible in the near future. If so, the only current possibility of distinguishing between various theoretical observables might be by deduction from more refined experimental results of the sorts that are now done.

It should also be noted that the Barnett and Pegg approach seems to yields an asymptotic behaviour of $1/2R^2$ for the variance of their phase operator candidates. It will be argued in our companion paper [19] that this result places too little weight on the high photon number states. Our argument will be that X_s represents an apparatus to measure the phase operator in an approximate sense. The weighting of the high photon number states is part of the limitations of the measurement apparatus. That a different result from ours then obtains is therefore not surprising, particularly in view of the fact that making various approximations to our formula also gives this alternate result.

For example, if the series for $\|\Delta(\varphi)\Phi_R\|$ is truncated, we come to consider the function

$$\mathcal{O}_N(R) = \frac{R^4}{\pi^2}\sum_{n=0}^{N}\frac{(2R^2)^n}{n!}|V_n(R)| \tag{6.37a}$$

for any $N \geqslant 0$. From our work above we may deduce that this has the asymptotic behaviour

$$\mathcal{O}_N(R) \sim \frac{1}{2R^2} \qquad R \to \infty. \tag{6.37b}$$

Alternatively, if we threw away the terms in V_n involving $\Gamma(\frac{1}{2}n, R^2)$, we would have the result

$$\frac{1}{4}\sum_{n\geqslant 1}\frac{2^n}{n!}R^{-2n}\gamma(n, R^2)^2 = O\left(\frac{1}{R^2}\right) \qquad R\to\infty \tag{6.38a}$$

which we obtained above. While the exact asymptotic of this series is not clear, it is true that

$$\frac{1}{4}\sum_{n\geqslant 1}\frac{2^n}{n!}R^{-2n}\gamma(n, k^2R^2)^2 \sim \frac{1}{2R^2} \qquad R\to\infty \tag{6.38b}$$

for any $0 < k < 1$. The point of this is that if we define a cut-off angle function by

$$\varphi_{R,k}(p,q) = \begin{cases} \varphi(p,q) & \text{if } (p-R)^2+q^2 < k^2R^2 \\ 0 & \text{otherwise} \end{cases} \tag{6.39a}$$

then

$$\|\Delta(\varphi_{R,k})\Phi_R\| = \tfrac{1}{4}\sum_{n\geqslant 1}\frac{2^n}{n!}R^{-2n}\gamma(n, k^2R^2)^2. \tag{6.39b}$$

There is no obvious physical justification for terminating the infinite series—for example, evaluating O_N does not correspond to truncating Φ_R to the first $N+1$ terms in its expansion in terms of the Hermite functions—or for restricting the domain of definition of $\Delta(\varphi)$ in the above way. It seems to us that there is a great deal of heuristic argument in the literature which is based on approximations such as these, and the corresponding statements about what sort of asymptotic behaviour ought to be expected are somewhat suspect. When the experimental situation is clearer, we may have a better idea of what really should be expected. Moreover, it may be that forcing through the calculational complexities of a sound theory will result in predicting new phenomena, which, after all, would be much more interesting.

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